Moments Defined by Subdivision Curves

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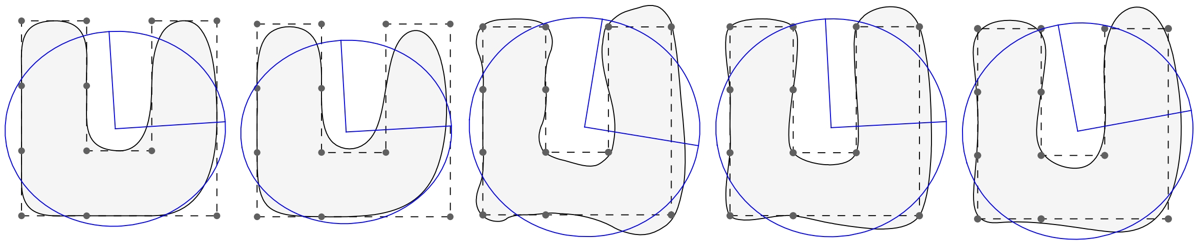

Figure: We compute the exact area, centroid, and inertia of the 2-dimensional sets bounded by subdivision curves. The illustration shows the principle axes of the inertia tensor drawn at the centroid of the area; five different subdivision schemes are used to demonstrate the universality of our derivation. ■

Abstract

We derive the $(d+2)$-linear forms that compute the moment of degree d of the area enclosed by a subdivision curve in the plane. We circumvent the need to solve integrals involving the basis function by exploiting a recursive relation and calibration that establishes the coefficients of the form within the nullspace of a matrix.

For demonstration, we apply the technique to the dual three-point scheme, the interpolatory C^1 four-point scheme, and the dual C^2 four-point scheme.

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Introduction

Subdivision of curves is a refinement procedure S for polygons. The algorithm is designed so that when applied iteratively, the increasingly dense point cycle converges to a piecewise smooth curve.

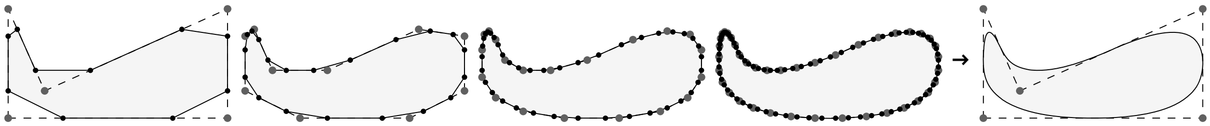


Figure: Several rounds of Chaikin's subdivision for a simple initial polygon P. The first iterations $P \to S(P)$, $S(P) \to S^2(P)$, ... are visualized. Right, the smooth limit curve $S^\infty(P)$ with input P as reference. ■

Our article is restricted to subdivision curves that are generated from polygons with finite number of control points $P = \left((\mathrm{px}_k, \mathrm{py}_k) \in \mathbb{R}^2 : k = 1, 2, \ \ldots, n\right)$ from the 2-dimensional plane. If the resulting curve $S^\infty(P)$ is compact, piecewise smooth, and not self-intersecting, we denote with $\Omega \subset \mathbb{R}^2$ the interior of the curve with $\partial\Omega = S^\infty(P)$. Then, the moment of degree $p + q = d$ for $p, q \in \{0, 1, 2, \ \ldots\}$ of the set Ω with respect to the x- and y-axis is well defined by the integral

$$M_{p,q}(\Omega) = \int_\Omega x^p \, y^q \, dx \, dy$$

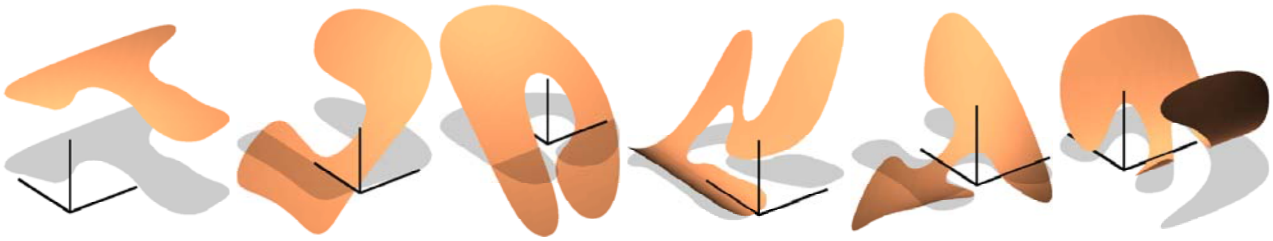


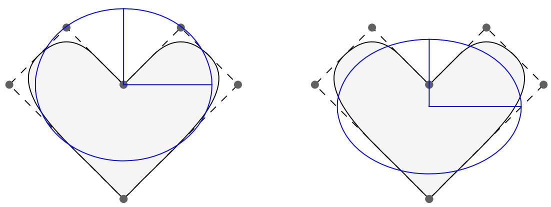
Figure: The monomials $1, x, y, x^2, x\,y, y^2$ over different domains $\Omega \subset \mathbb{R}^2$ bounded by subdivision curves. ■

The moments for small degrees have interpretation as

$$\text{area}(\Omega) = M_{0,0}(\Omega) \qquad (d = 0)$$

$$\text{centroid}(\Omega) = \frac{1}{\text{area}(\Omega)} \left(M_{1,0}(\Omega), M_{0,1}(\Omega)\right) \qquad (d = 1)$$

$$\text{inertia}(\Omega) = (M_{2,0}(\Omega), M_{1,1}(\Omega), M_{0,2}(\Omega)) \qquad (d = 2)$$



Example: Chaikin's subdivision applied to $P = ((0, -1), (0, -1), (1, 0), (0.5, 0.5), (0, 0), (0, 0), (-0.5, 0.5), (-1, 0))$ results in a curve that bounds an $\text{area}(\Omega) = 11/8$ with $\text{centroid}(\Omega) = (0, -21/110)$,

$$\text{inertia}(\Omega) = \frac{1}{8960} (2161, 0, 1607), \text{ and } \text{inertia}(\Omega - \text{centroid}(\Omega)) = \frac{1}{492\,800} (118\,855, 0, 63\,689).$$

Eigenvalue decomposition of the inertia tensor gives the principal axes of the ellipsoid with equivalent inertia. ■

The moments derived in the article have diverse applications: 1) The formulas allow to design subdivision curves with exact area, centroid, and inertia. 2) By translation of control points, curves can be deformed subject to preservation of moments. 3) The set bounded by a subdivision curve extruded along the interval $[a, b] \subset \mathbb{R}$ of the z-axis has moment of

$$M_{p,q,r}(\Omega \times [a, b]) = \int_{\Omega \times [a,b]} x^p\, y^q\, z^r \, dx\, dy\, dz = M_{p,q}(\Omega) \int_{[a,b]} z^r \, dz \qquad \text{for } p, q, r = 0, 1, 2, \ldots$$

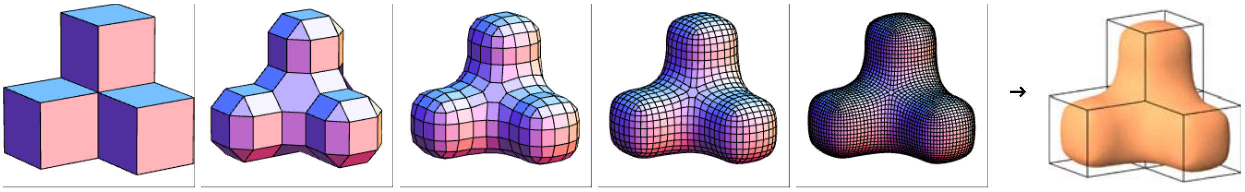
4) Countless computer games use planar graphics and physics. If a subdivision curve is the contour of an animated entity, our formulas help to make the motion more plausible. However, the term $M_{p,q}(\Omega)$ assumes constant mass density across Ω.

Previous work

If the boundary $\partial\Omega$ has a known piecewise parameterization by polynomials, the moment $M_{p,q}(\Omega)$ is computed by integration using the divergence theorem. The methodology applies to sets bounded by B-spline subdivision curves, including Chaikin's scheme, as well as Bézier patches, see [Gonzalez/McCammon/Peters 1998].

[Warren/Weimer 2002] compute the bilinear form for the area enclosed by curves generated by the interpolatory C^1 four-point scheme with tension parameter $\omega = 1/16$ through mojo.

[Hakenberg et al. 2014] derive the trilinear forms that compute the volume enclosed by subdivision surfaces, i.e. the moment of degree $d = 0$ of a 3-dimensional set. The authors conclude that moments of higher degree $d = 1, 2, \ldots$ of the sets bounded by subdivision surfaces are not tractable by today's computational means due to the large number of unknown coefficients in the $(d + 3)$-linear forms. Therefore, we restrict the concept to the simpler, 2-dimensional case of subdivision curves.



Example: Surface subdivision applied to a simple initial mesh of 4 unit cubes glued together. The limit surface bounds a set of volume $\frac{10\,357\,799\,098\,161+2\,535\,566\,756\,\sqrt{5}}{3\,238\,292\,736\,000}$. The centroid is *not* known explicitly, however. ■

Overview

The contribution of this article is a formalism to derive the moments $M_{p,q}(\Omega)$ for sets bounded by subdivision curves that do not have a closed-form parameterization. The moment depends on the subdivision scheme S and the initial polygon P. We derive the formula using the conceptual approach

$$M_{p,q}(\Omega) = M_{p,q}(\underbrace{S^{\infty}(P)}_{=\partial\Omega}) = M_{p,q}(P)$$

The first equality is established through the divergence theorem. The second equality is the result of identifying an operator $M_{p,q}$ that is

1) invariant under one round of subdivision $M_{p,q}(P) = M_{p,q}(S(P))$, and

2) reproduces the correct momentum value for a known special case, for instance the unit square $\Omega = [0, 1]^2$.

Once the formulas are clarified, the equation serves as a definition for the $M_{p,q}$ operator overloading. $M_{p,q}(P)$ is always interpreted with a specific subdivision scheme S in mind.

Our article is structured as follows. First, we recap the basics of curve subdivision: the basis function of a scheme, and refinement matrices. Chaikin's scheme serves as an example. Then, we derive the formula for $M_{p,q}(P)$ for binary, stationary subdivision schemes. We demonstrate the practicability of our formalism on several popular schemes. The computation of moment values defined by a number of simple example curves serves as a reference for alternative implementations.

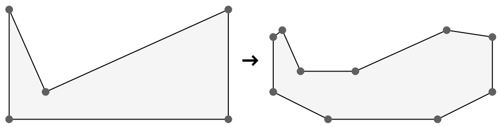
Binary Subdivision

The first subdivision scheme for curves was published by [Chaikin 1974]. His work inspired not only the discovery of other polygon refinement algorithms but also the development of surface subdivision. Chaikin described his method as "*a fast algorithm for the generation of arbitrary curves*". And further, "*the algorithm is recursive, using only integer addition, one-bit right shifts, complementation and comparisons, and produces a sequential list of raster points which constitute the curve*". One round of subdivision introduces two points with coordinates

$$\left(\frac{3}{4}\,px_k + \frac{1}{4}\,px_{k+1},\ \frac{3}{4}\,py_k + \frac{1}{4}\,py_{k+1}\right), \text{ and } \left(\frac{1}{4}\,px_k + \frac{3}{4}\,px_{k+1},\ \frac{1}{4}\,py_k + \frac{3}{4}\,py_{k+1}\right)$$

for all $k = 1, 2, \ldots, n$. The two affine linear combinations that determine the points in $S(P)$ along each edge of P are more conveniently depicted as

The coefficients are applied coordinatewise and refered to as *weights*. The points of P are also refered to as *control points*. Since P constitutes a closed polygon, the sequence of n control points is interpreted as a cycle. The index k is understood modulo n. For instance, index $k = 0$ is identified with $k = n$.



Example: The “whale” contour is the cycle $P = \left((0, 0), (2, 0), (2, 1), \left(\frac{1}{3}, \frac{1}{4}\right), (0, 1)\right)$. One round of Chaikin subdivision results in $S(P) = \left(\left(\frac{1}{2}, 0\right), \left(\frac{3}{2}, 0\right), \left(2, \frac{1}{4}\right), \left(2, \frac{3}{4}\right), \left(\frac{19}{12}, \frac{13}{16}\right), \left(\frac{3}{4}, \frac{7}{16}\right), \left(\frac{1}{4}, \frac{7}{16}\right), \left(\frac{1}{12}, \frac{13}{16}\right), \left(0, \frac{3}{4}\right), \left(0, \frac{1}{4}\right)\right)$. ■

The *basis function* $\varphi : \mathbb{R} \to \mathbb{R}$ of a subdivision scheme has compact support in the interval $[0, m] \subset \mathbb{R}$ for an integer m. The function φ parameterizes the limit curve $S^\infty(P)$ piecewise as

$$c_k(t) = \sum_{i=1}^{m} \varphi(t - i + m)\,(\mathrm{px}_{k+i}, \mathrm{py}_{k+i}) = \sum_{i=1}^{m} b_i(t)\,(\mathrm{px}_{k+i}, \mathrm{py}_{k+i}) \qquad \text{for } t \in D = [0, 1], \text{ and } k = 1, 2, \ldots, n.$$

The basis function segment $b_i : D \to \mathbb{R}$ is defined as $b_i(t) = \varphi(t - i + m)$ for $i = 1, 2, \ldots, m$. The segments b_i form a partition of unity $\sum_{i=1}^{m} b_i(t) = 1$ for all $t \in D$.

The curve segment $c_k : D \to \mathbb{R}^2$ with $c_k(t) = (\mathrm{cx}_k(t), \mathrm{cy}_k(t))$ is refered to as *facet* f^k for $k = 1, 2, \ldots, n$. The curve $S^\infty(P)$ is the union of the n facets. The integer m is the number of control points that determine a single facet. A facet parameterized by c_k for $k = 1, 2, \ldots, n$ depends on the control points $(\mathrm{px}_{k+i}, \mathrm{py}_{k+i})$ for $i = 1, 2, \ldots, m$. The notion of facets is central to our approach, since the global moment is computed as a sum over the moments spanned by the facets.

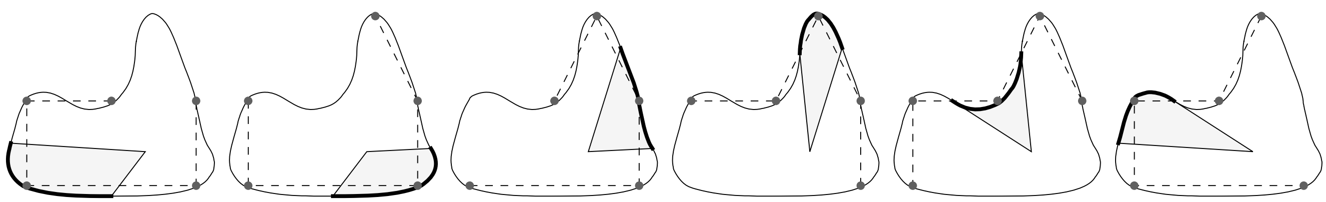
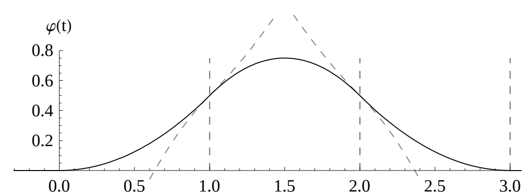


Figure: The set bounded by a subdivision curve $S^\infty(P)$ as the union of conic sets spanned by each facet f^k for $k = 1, 2 \ldots, n$. In the illustration, the $m = 5$ control points closest to f^k determine the shape of the facet. ■

The basis function φ characteristic to Chaikin’s scheme is the piecewise polynomial function

$$\varphi(t) = \begin{cases} \frac{1}{2} t^2 & t \in [0, 1] \\ \frac{1}{2}\left(-3 + 6t - 2t^2\right) & t \in [1, 2] \\ \frac{1}{2}(3 - t)^2 & t \in [2, 3] \end{cases}$$



and $\varphi(t) = 0$ for $t \notin [0, 3]$ outside the support. The basis function segments are the quadratic polynomials

$$b_1(t) = \frac{1}{2}(1 - t)^2,\; b_2(t) = \frac{1}{2}\left(1 + 2t - 2t^2\right), \text{ and } b_3(t) = \frac{1}{2} t^2 \qquad \text{for } t \in D = [0, 1].$$

For a general scheme, the basis function φ does not have a closed-form expression. The graph of φ is the limit curve $S^\infty(P_\delta)$ of the infinite point sequence $P_\delta = \{(z + m/2, \delta_{z,0}) : z \in \mathbb{Z}\}$.

For the derivation of moments defined by subdivision curves, we do not require $b_i : D \to \mathbb{R}$ to have a closed-form expression. To establish the aforementioned invariance of $M_{p,q}(P) = M_{p,q}(S(P))$, we use that the collection of functions b_i for $i = 1, 2, \ldots, m$ is *refinable* with respect to the split of the domain $D = [0, 1]$ into $D_1 = \left[0, \frac{1}{2}\right]$, and $D_2 = \left[\frac{1}{2}, 1\right]$: Let affine-linear maps $T_h : D_h \to D$ for $h \in \{1, 2\}$ be defined as $T_1(s) = 2s$, and $T_2(s) = 2s - 1$. Then, there exist matrices S^1, and S^2 with dimensions $m \times m$ that satisfy

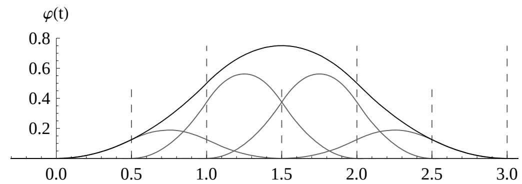
$$b_i(s) = \sum_{a=1}^{m} b_a(T_h(s))\, S^h_{a,i} \qquad \text{for all } i = 1, 2, \ldots, m,\; s \in D_h, \text{ and } h \in \{1, 2\}.$$

The matrix S^h maps the m control points of f coordinatewise to those of f^h as

$$\left(\left(\sum_{j=1}^{m} S^h_{i,j}\,\mathrm{px}_j,\ \sum_{j=1}^{m} S^h_{i,j}\,\mathrm{py}_j\right) : \text{for } i = 1, 2, \ \ldots, m\right) \qquad \text{for } h \in \{1, 2\}.$$

For Chaikin’s scheme, the 3×3 matrices are

$$S^1 = \frac{1}{4}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \text{ and } S^2 = \frac{1}{4}\begin{pmatrix} 1 & 3 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$



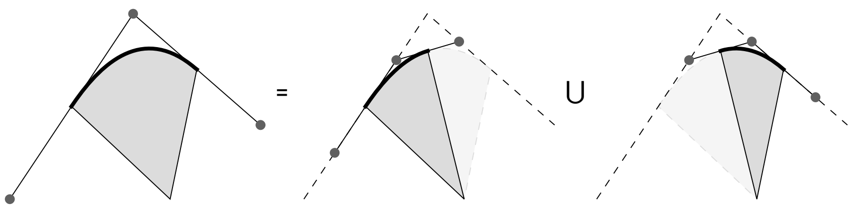


Figure: The refinement of a facet f into two smaller facets f_1 and f_2 through one iteration of Chaikin’s subdivision. The illustration includes the $m = 3$ control points that determine each facet. The matrix S^h maps the control points of f coordinatewise to those of f_h for $h \in \{1, 2\}$. ■

Derivation of Moments

Divergence Theorem

Propaedeutic: Let $\gamma : [0, n] \subset \mathbb{R} \to \mathbb{R}^2$ be a piecewise smooth curve $\gamma(t) = (\gamma_x(t), \gamma_y(t))$ that parameterizes the boundary of a compact set $\Omega \subset \mathbb{R}^2$, i.e. $\gamma([0, n]) = \partial\Omega$. The tangent vector is $\mathrm{d}\gamma(t) = (\partial_t \gamma_x(t), \partial_t \gamma_y(t))$ with perpendicular $\mathrm{d}\gamma(t)^\perp = (\partial_t \gamma_y(t), -\partial_t \gamma_x(t))$, and unit normal $\vec{n} = \mathrm{d}\gamma(t)^\perp / \|\mathrm{d}\gamma(t)^\perp\|$. The vectors have the same length, i.e. $\|\mathrm{d}\gamma(t)^\perp\| = \|\mathrm{d}\gamma(t)\|$ for all $t \in [0, n]$. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable vector field. Then, the divergence theorem states that

$$\int_\Omega \operatorname{div} G \, dx \, dy = \int_{\partial\Omega} G \cdot \vec{n} \, d\partial\Omega = \int_{[0,n]} G(\gamma(t)) \cdot \frac{\mathrm{d}\gamma(t)^\perp}{\|\mathrm{d}\gamma(t)^\perp\|} \|\mathrm{d}\gamma(t)\| \, dt = \int_{[0,n]} G(\gamma(t)) \cdot \mathrm{d}\gamma(t)^\perp \, dt. \ ■$$

We apply the equality to monomials over $\Omega \subset \mathbb{R}^2$ and subdivision curves that parameterize the boundary $\partial\Omega$ by a sequence of smooth facets $c_k(t) = (\mathrm{cx}_k(t), \mathrm{cy}_k(t))$ with perpendicular $\mathrm{d}c_k(t)^\perp = (\partial_t \mathrm{cy}_k(t), -\partial_t \mathrm{cx}_k(t))$ for $t \in D = [0, 1]$, and $k = 1, 2, \ \ldots, n$. As vector field $G_{p,q} : \mathbb{R}^2 \to \mathbb{R}^2$, we choose $G_{p,q}(x, y) = \left(\frac{1}{p+1} x^{p+1} y^q, 0\right)$ for $p, q \in \{0, 1, 2, \ \ldots\}$ with $\operatorname{div} G_{p,q} = x^p y^q$. Then,

$$M_{p,q}(\Omega) = \int_\Omega x^p y^q \, dx \, dy = \sum_{k=1}^{n} \int_D G_{p,q}(c_k(t)) \cdot \mathrm{d}c_k(t)^\perp \, dt = \sum_{k=1}^{n} \frac{1}{p+1} \int_D \mathrm{cx}_k(t)^{p+1} \, \mathrm{cy}_k(t)^q \, \partial_t \mathrm{cy}_k(t) \, dt.$$

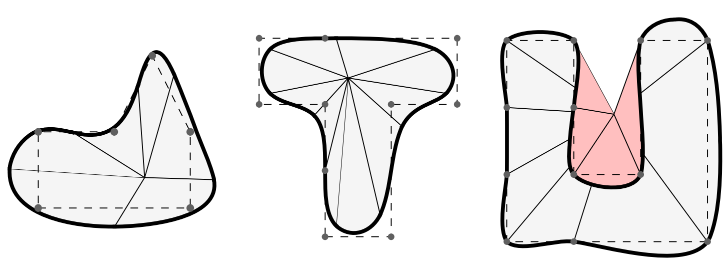


Figure: Decomposition of Ω into n conic areas. The boundary $\partial\Omega$ is partitioned into the facets f^k for $k = 1, 2, \ \ldots, n$. The integral over the red area cancels the excess. ■

Multilinear Form

The moment $M_{p,q}(\Omega)$ is a sum over all facets. It suffices to investigate the term corresponding to the contribution of a single facet. To keep the notation to the point, we temporarily drop the index k. Then, the coordinate functions become $\mathrm{cx}(t) = \sum_{i=1}^{m} b_i(t)\, \mathrm{px}_i$, and $\mathrm{cy}(t) = \sum_{i=1}^{m} b_i(t)\, \mathrm{py}_i$ for $t \in D$. To simplify further, we use $\sum_{i_1,\ldots,i_r}^{m}$ as abbreviation for $\sum_{i_1=1}^{m} \ldots \sum_{i_r=1}^{m}$.

$$
\begin{aligned}
M_{p,q}(f) &= \frac{1}{p+1} \int_D \mathrm{cx}(t)^{p+1}\, \mathrm{cy}(t)^q\, \partial_t\, \mathrm{cy}(t)\, dt \\
&= \frac{1}{p+1} \int_D \Big(\sum_{i=1}^{m} b_i(t)\, \mathrm{px}_i\Big)^{p+1} \Big(\sum_{j=1}^{m} b_j(t)\, \mathrm{py}_j\Big)^q \Big(\sum_{j=1}^{m} \partial_t b_j(t)\, \mathrm{py}_j\Big)\, dt \\
&= \frac{1}{p+1} \sum_{i_1,\ldots,i_{p+1}}^{m} \sum_{j_1,\ldots,j_q,j_{q+1}}^{m} \int_D b_{i_1}(t) \ldots b_{i_{p+1}}(t)\, b_{j_1}(t) \ldots b_{j_q}(t) \left(\partial_t b_{j_{q+1}}(t)\right) dt\, \mathrm{px}_{i_1} \ldots \mathrm{px}_{i_{p+1}}\, \mathrm{py}_{j_1} \ldots \mathrm{py}_{j_q}\, \mathrm{py}_{j_{q+1}} \\
&= \frac{1}{p+1} \sum_{i_1,\ldots,i_{p+1},j_1,\ldots,j_{q+1}}^{m} \overline{A}^{(d)}_{i_1,\ldots,i_{p+1},j_1,\ldots,j_{q+1}}\, \mathrm{px}_{i_1} \ldots \mathrm{px}_{i_{p+1}}\, \mathrm{py}_{j_1} \ldots \mathrm{py}_{j_{q+1}}
\end{aligned}
$$

The final expression shows that $M_{p,q}(f)$ is a $(d+2)$-linear form in the m points of P that determine the facet f.

The coefficients of the form are universal for any combination p, q with $p+q=d$ except for the leading factor $\frac{1}{p+1}$. A solution to the coefficients are the integrals

$$
\overline{A}^{(d)}_{i_1,\ldots,i_{d+1},i_{d+2}} := \int_D b_{i_1}(t) \ldots b_{i_{d+1}}(t)\, (\partial_t b_{i_{d+2}}(t))\, dt \qquad \text{for } i_1, \ldots, i_{d+1}, i_{d+2} \in \{1, 2, \ldots, m\}.
$$

Whenever the basis function segments $b_i : D \to \mathbb{R}$ are polynomials, straightforward evaluation of the integral expressions gives the multilinear form $\frac{1}{p+1} \overline{A}^{(d)}$ that determines the moment $M_{p,q}(f)$.

Recursion

Evaluating the integrals directly is not possible for a general subdivision scheme. However, we establish an implicit relation through reorganization of the integral expression. First, we split the domain $D = [0, 1]$ into $D_1 = \left[0, \frac{1}{2}\right]$, and $D_2 = \left[\frac{1}{2}, 1\right]$. Recall that b_i are refinable $b_i(s) = \sum_{a=1}^{m} b_a(T_h(s))\, S^h_{a,i}$ for $s \in D_h$, $i = 1, 2, \ldots, m$, and $h \in \{1, 2\}$. The change of coordinates is $T_h(s) = t$ with $T_h'(s)\, ds = 2\, ds = dt$ for $h \in \{1, 2\}$. Then, the recursive relation for the integrals is derived as

$$
\begin{aligned}
&\int_D b_{i_1}(t) \ldots b_{i_{d+1}}(t)\, (\partial_t b_{i_{d+2}}(t))\, dt \\
&= \sum_{h=1}^{2} \int_{D_h} b_{i_1}(s) \ldots b_{i_{d+1}}(s)\, (\partial_s b_{i_{d+2}}(s))\, ds \\
&= \sum_{h=1}^{2} \int_{D_h} \Big(\sum_{a_1}^{m} b_{a_1}(T_h(s))\, S^h_{a_1,i_1}\Big) \ldots \Big(\sum_{a_{d+1}}^{m} b_{a_{d+1}}(T_h(s))\, S^h_{a_{d+1},i_{d+1}}\Big) \Big(\sum_{a_{d+2}}^{m} \partial_s b_{a_{d+2}}(T_h(s))\, S^h_{a_{d+2},i_{d+2}}\Big)\, ds \\
&= \sum_{h=1}^{2} \int_D \Big(\sum_{a_1}^{m} b_{a_1}(t)\, S^h_{a_1,i_1}\Big) \ldots \Big(\sum_{a_{d+1}}^{m} b_{a_{d+1}}(t)\, S^h_{a_{d+1},i_{d+1}}\Big) \Big(2 \sum_{a_{d+2}}^{m} \partial_t b_{a_{d+2}}(t)\, S^h_{a_{d+2},i_{d+2}}\Big) \frac{1}{2}\, dt \\
&= \sum_{h=1}^{2} \int_D \Big(\sum_{a_1}^{m} b_{a_1}(t)\, S^h_{a_1,i_1}\Big) \ldots \Big(\sum_{a_{d+1}}^{m} b_{a_{d+1}}(t)\, S^h_{a_{d+1},i_{d+1}}\Big) \Big(\sum_{a_{d+2}}^{m} \partial_t b_{a_{d+2}}(t)\, S^h_{a_{d+2},i_{d+2}}\Big)\, dt \\
&= \sum_{h=1}^{2} \sum_{a_1,\ldots,a_{d+1},a_{d+2}}^{m} \int_D b_{a_1}(t) \ldots b_{a_{d+1}}(t)\, (\partial_t b_{a_{d+2}}(t))\, dt\, S^h_{a_1,i_1} \ldots S^h_{a_{d+1},i_{d+1}}\, S^h_{a_{d+2},i_{d+2}}
\end{aligned}
$$

We substitute the unknown integral expressions with the coefficients $A^{(d)}_{i_1,\ldots,i_{d+1},i_{d+2}}$ of a general tensor $A^{(d)}$ of rank $d+2$ and compress the equations to

$$
A^{(d)}_{i_1,\ldots,i_{d+1},i_{d+2}} = \sum_{h=1}^{2} \sum_{a_1,\ldots,a_{d+1},a_{d+2}}^{m} A^{(d)}_{a_1,\ldots,a_{d+1},a_{d+2}}\, S^h_{a_1,i_1} \ldots S^h_{a_{d+1},i_{d+1}}\, S^h_{a_{d+2},i_{d+2}} \qquad \text{for all } i_1, \ldots, i_{d+1}, i_{d+2} \in \{1, 2, \ldots, m\}.
$$

Among the multilinear forms $A^{(d)}$ that satisfy the relation, some compute the valid moment $M_{p,q}(f)$ and are a substitute for $\overline{A}^{(d)}$.

The relation is equivalent to the invariance of the moment formula under one round of subdivision, i.e. demanding $M_{p,q}(f) = M_{p,q}(f_1) + M_{p,q}(f_2)$. Then, $M_{p,q}(P) = M_{p,q}(S(P))$ follows easily.

For the following consolidation, we enumerate the multi-index $(i_1, i_2, \ldots, i_{d+2})$ in a linear fashion

$$\sharp(i_1, i_2, \ldots, i_{d+2}) = i_1 + (i_2 - 1)\, m + \ldots + (i_{d+2} - 1)\, m^{d+1}.$$

Corollary 1: The coefficients of the tensor $A^{(d)}$ satisfy the homogeneous linear system $(E - I).x = 0$ where

$$E_{\sharp(i_1,\ldots,i_{d+1},i_{d+2}),\sharp(a_1,\ldots,d_{d+1},a_{d+2})} = \sum_{h=1}^{2} S^h_{a_1,i_1} \ldots S^h_{a_{d+1},i_{d+1}}\, S^h_{a_{d+2},i_{d+2}} \qquad \text{for all } i_1, \ldots, i_{d+2}, a_1, \ldots, a_{d+2} \in \{1, 2, \ldots, m\},$$

and I is the identity matrix. x is the vector with $x_{\sharp(i_1,\ldots,i_{d+2})} = A^{(d)}_{i_1,\ldots,i_{d+1},i_{d+2}}$. A solution x is an element in the nullspace of the matrix $E - I$. ■

We observe that the matrix E is the sum over $h \in \{1, 2\}$ of the $(d+2)$-fold Kronecker-product of S^h transposed. The coefficients in E only depend on the subdivision weights encoded in the matrices S^h.

Symmetry

The integral $\overline{A}^{(d)}_{i_1,\ldots,i_{d+1},i_{d+2}}$ is invariant under permutation of the first $d+1$ factors $b_{i_1}(t), \ldots, b_{i_{d+1}}(t)$. That means, the indices can be arranged to be non-decreasing $i_1 \leq i_2 \leq \ldots \leq i_{d+1}$. The last factor $\partial b_{i_{d+2}}(t)$ is left alone. The integral $\overline{A}^{(d)}_{i_1,\ldots,i_{d+1},i_{d+2}}$ guarantees that there exists a non-trivial tensor solution $A^{(d)}$ to Corollary 1 with that symmetry. We may demand

$$A^{(d)}_{i_1,\ldots,i_{d+1},i_{d+2}} = A^{(d)}_{\mathrm{sort}(i_1,\ldots,i_{d+1}),i_{d+2}} \qquad \text{for all } i_1, \ldots, i_{d+1}, i_{d+2} \in \{1, 2, \ldots, m\}.$$

The equivalence reduces the number of variables from m^{d+2} down to $\binom{d+m}{d+1} m$.

Remark: Typically, the basis function satisfies $\varphi(t) = \varphi(m - t)$ for $t \in \mathbb{R}$. Then, $\varphi'(t) = -\varphi'(m - t)$ follows. These symmetries extend to the basis function segments as $b_i(t) = b_{\sigma(i)}(1 - t)$, and $b_i'(t) = -b_{\sigma(i)}'(1 - t)$ where $\sigma(i) := m - i + 1$ for all $t \in D = [0, 1]$, and $i = 1, 2, \ldots, m$. Reparametrization of the domain D of the integral shows

$$\overline{A}^{(d)}_{i_1,\ldots,i_{d+1},i_{d+2}} = -\overline{A}^{(d)}_{\sigma(i_1),\ldots,\sigma(i_{d+1}),\sigma(i_{d+2})} \qquad \text{for all } i_1, \ldots, i_{d+1}, i_{d+2} \in \{1, 2, \ldots, m\}.$$

The last index i_{d+2} may be confined to $i_{d+2} = 1, 2, \ldots, \lceil m/2 \rceil$, for instance. ■

Calibration

The solution space of Corollary 1, i.e. the nullspace of matrix $E - I$, has to be truncated to a subspace of multilinear forms that result in the correct moment value. We refer to this procedure as *calibration*. We reintroduce the index $k = 1, 2, \ldots, n$ to enumerate the facets. The moment of facet f^k is the term

$$M_{p,q}(f^k) = \frac{1}{p+1} \sum_{i_1,\ldots,i_{p+1}}^{m} \sum_{j_1,\ldots,j_{q+1}}^{m} A^{(d)}_{i_1,\ldots,i_{p+1},j_1,\ldots,j_{q+1}}\, \mathrm{px}_{k+i_1} \ldots \mathrm{px}_{k+i_{p+1}}\, \mathrm{py}_{k+j_1} \ldots \mathrm{py}_{k+j_{q+1}}$$

The global moment is $M_{p,q}(P) = \sum_{k=1}^{n} M_{p,q}(f^k)$. We confine the solution space using a simple, special case:

Exercise 1: For the set $\Omega = [0, 1]^2 \subset \mathbb{R}^2$, i.e. the unit square aligned at the origin of the plane, the moment is

$$M_{p,q}([0, 1]^2) = \int_{[0,1]} \int_{[0,1]} x^p\, y^q\, dx\, dy = \int_{[0,1]} x^p\, dx \int_{[0,1]} y^q\, dy = \frac{1}{p+1}\, \frac{1}{q+1} \qquad \text{for } 0 \leq p, q. \; ■$$

For a subdivision scheme S that has basis function φ with support $[0, m] \subset \mathbb{R}$, the point cycle

$$P_c = \Big(\underbrace{(0, 0), \ldots, (0, 0)}_{m-1 \text{ times}}, \underbrace{(1, 0), \ldots, (1, 0)}_{m-1 \text{ times}}, \underbrace{(1, 1), \ldots, (1, 1)}_{m-1 \text{ times}}, \underbrace{(0, 1), \ldots, (0, 1)}_{m-1 \text{ times}} \Big)$$

results in a limit curve $S^\infty(P_c)$ that contains the boundary of the set $\Omega = [0, 1]^2$. If all subdivision weights are non-negative, $S^\infty(P_c) = \partial([0, 1]^2)$ precisely, otherwise there is an overshoot of the curve that does not contribute to the moment however.

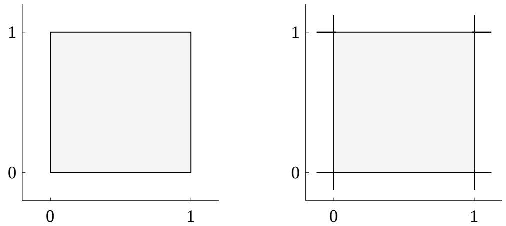


Figure: Left, $S^{\infty}(P_c)$ for a scheme with non-negative weights. Right, overshoot due to negative weights. ■

Calibration demands that elements $A^{(d)}$ from the nullspace of $E - I$ additionally satisfy $M_{p,q}(P_c) = \frac{1}{p+1}\frac{1}{q+1}$.

[Warren/Weimer 2002] also employed the described approach. We quote from page 166: "*For the* [interpolatory C^1] *four-point scheme, this condition requires the use of five-fold points at each of the corners of the square. (The factor of five is due to the size of the mask associated with the four-point scheme.)*"

In all examples that we encountered, calibration does not result in a unique solution. Instead, a form $A^{(d)}$ has to be selected from a 1-dimensional vectorspace. The choice of the extra parameter affects the contribution of a single facet $M_{p,q}(f^k)$ but is canceled in the global sum $M_{p,q}(P) = \sum_{k=1}^{n} M_{p,q}(f^k)$.

Summary

We state the formulas for moments of low degree for a single facet.

($d = 0$): The bilinear form $A^{(0)}$ has dimensions $m \times m$ and gives

$$M_{0,0}(f) = \frac{1}{1} \sum_{i_1,j_1}^{m} A^{(0)}_{i_1,j_1}\, \mathrm{px}_{i_1}\, \mathrm{py}_{j_1} \qquad (p = 0,\ q = 0)$$

($d = 1$): The trilinear form $A^{(1)}$ has dimensions $m \times m \times m$ and gives

$$M_{1,0}(f) = \frac{1}{2} \sum_{i_1,i_2,j_1}^{m} A^{(1)}_{i_1,i_2,j_1}\, \mathrm{px}_{i_1}\, \mathrm{px}_{i_2}\, \mathrm{py}_{j_1} \qquad (p = 1,\ q = 0)$$

$$M_{0,1}(f) = \frac{1}{1} \sum_{i_1,j_1,j_2}^{m} A^{(1)}_{i_1,j_1,j_2}\, \mathrm{px}_{i_1}\, \mathrm{py}_{j_1}\, \mathrm{py}_{j_2} \qquad (p = 0,\ q = 1)$$

($d = 2$): The 4-linear form $A^{(2)}$ gives

$$M_{2,0}(f) = \frac{1}{3} \sum_{i_1,i_2,i_3,j_1}^{m} A^{(2)}_{i_1,i_2,i_3,j_1}\, \mathrm{px}_{i_1}\, \mathrm{px}_{i_2}\, \mathrm{px}_{i_3}\, \mathrm{py}_{j_1} \qquad (p = 2,\ q = 0)$$

$$M_{1,1}(f) = \frac{1}{2} \sum_{i_1,i_2,j_1,j_2}^{m} A^{(2)}_{i_1,i_2,j_1,j_2}\, \mathrm{px}_{i_1}\, \mathrm{px}_{i_2}\, \mathrm{py}_{j_1}\, \mathrm{py}_{j_2} \qquad (p = 1,\ q = 1)$$

$$M_{0,2}(f) = \frac{1}{1} \sum_{i_1,j_1,j_2,j_3}^{m} A^{(2)}_{i_1,j_1,j_2,j_3}\, \mathrm{px}_{i_1}\, \mathrm{py}_{j_1}\, \mathrm{py}_{j_2}\, \mathrm{py}_{j_3} \qquad (p = 0,\ q = 2)$$

The global moment is a sum over all facets.

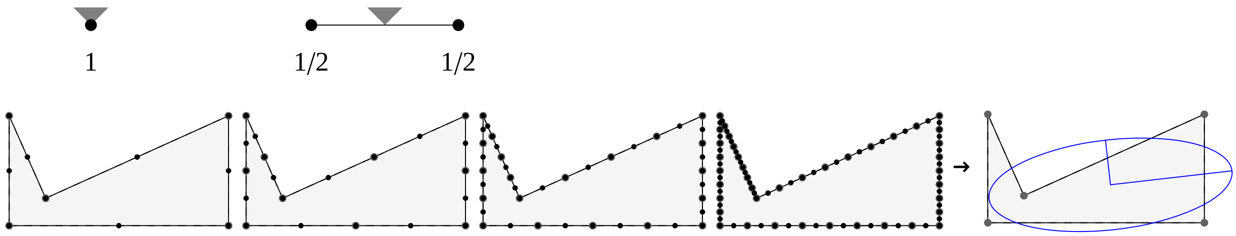
Applications

We apply the formalism to several relevant subdivision algorithms. The particular schemes that we treat have support $m = 2, 3, \ldots, 7, 10$. The complexity increases with m. In our examples for $m \in \{5, 6, 7, 10\}$, the characteristic basis functions do not have a closed-form expression.

The source code that solves and calibrates the multilinear forms $A^{(d)}$ for a given subdivision scheme S is available at [Hakenberg 2014]. The web resource also lists additional example curves with corresponding moment values of low degrees for verification of alternative implementations of the formulas.

Linear B-Spline Scheme

Linear subdivision is vertex interpolation and mid-edge insertion. The weights are



Example: Several iterations of linear subdivision. The moments are $\text{area}(\Omega) = 5/4$, $\text{centroid}(\Omega) = \left(\frac{17}{15}, \frac{7}{20}\right)$, and $\text{inertia}(\Omega - \text{centroid}(\Omega)) = \left(\frac{167}{360}, \frac{11}{240}, \frac{131}{1920}\right)$. ■

A facet is the linear, convex interpolation between two adjacent points

$$c_k(t) = (1 - t)\,(\text{px}_{k+1}, \text{py}_{k+1}) + t(\text{px}_{k+2}, \text{py}_{k+2}) \qquad \text{for } t \in D = [0, 1], \text{ and } k = 1, 2, \ldots, n.$$

Consequently, a facet is determined by $m = 2$ control points. The subdivision curve $S^\infty(P)$ is a polygon. The refinement matrices are

$$S^1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \text{ and } S^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

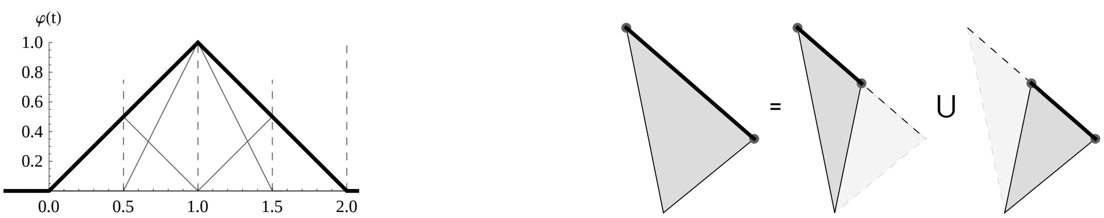


Figure: The basis function segments are $b_1(t) = 1 - t$, and $b_2(t) = t$ for $t \in D = [0, 1]$. Right, the matrix S^h maps the control points of f coordinatewise to those of f_h for $h \in \{1, 2\}$. ■

($d = 0$): The bilinear form $A^{(0)}$ after calibration is

$$A^{(0)} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \lambda\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for arbitrary } \lambda \in \mathbb{R}.$$

For $\lambda = 0$, the formula for the area of a polygon $P = \left((\text{px}_k, \text{py}_k) \in \mathbb{R}^2 : k = 1, 2, \ldots, n\right)$ is the familiar expression

$$M_{0,0}(P) = \frac{1}{2}\sum_{k=1}^{n} \det\begin{pmatrix} \text{px}_k & \text{py}_k \\ \text{px}_{k+1} & \text{py}_{k+1} \end{pmatrix} = \frac{1}{2}\sum_{k=1}^{n} (\text{px}_k\,\text{py}_{k+1} - \text{px}_{k+1}\,\text{py}_k) = \frac{1}{2}\sum_{k=1}^{n} \text{px}_k(\text{py}_{k+1} - \text{py}_{k-1}).$$

($d = 1$): We detail the derivation of the trilinear form $A^{(1)}$ for the moment of degree 1: Corollary 1 specifies the homogeneous linear system as

$$(E - I).x = \frac{1}{8}\begin{pmatrix} 1 & 4 & 4 & 2 & 4 & 2 & 2 & 1 \\ 1 & -2 & 0 & 2 & 0 & 2 & 0 & 1 \\ 1 & 0 & -2 & 2 & 0 & 0 & 2 & 1 \\ 1 & 2 & 2 & -2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -2 & 2 & 2 & 1 \\ 1 & 2 & 0 & 0 & 2 & -2 & 0 & 1 \\ 1 & 0 & 2 & 0 & 2 & 0 & -2 & 1 \\ 1 & 2 & 2 & 4 & 2 & 4 & 4 & 1 \end{pmatrix}.x = 0$$

with $x = \left(A^{(1)}_{1,1,1}, A^{(1)}_{2,1,1}, A^{(1)}_{1,2,1}, A^{(1)}_{2,2,1}, A^{(1)}_{1,1,2}, A^{(1)}_{2,1,2}, A^{(1)}_{1,2,2}, A^{(1)}_{2,2,2}\right)$.

Approach 1 The nullspace of matrix $E - I$ is 3-dimensional and spanned by

$$A^{(1)}_{..,1} = \begin{pmatrix} -\lambda_1 & -\lambda_2 + \lambda_3 \\ -\lambda_2 - \lambda_3 & -2\,\lambda_2 \end{pmatrix},\ A^{(1)}_{..,2} = \begin{pmatrix} 2\,\lambda_2 & \lambda_2 + \lambda_3 \\ \lambda_2 - \lambda_3 & \lambda_1 \end{pmatrix} \qquad \text{for } \lambda_1,\ \lambda_2,\ \lambda_3 \in \mathbb{R}.$$

For calibration, we equate the form applied to $P_c = ((0,\ 0),\ (1,\ 0),\ (1,\ 1),\ (0,\ 1))$ to the analytical result from Exercise 1. The requirements are

$$M_{1,0}(P_c) = 3\,\lambda_2 = \frac{1}{1+1}\,\frac{1}{1} = \frac{1}{2},\text{ and } M_{0,1}(P_c) = 3\,(\lambda_2 - \lambda_3) = \frac{1}{1}\,\frac{1}{1+1} = \frac{1}{2}.$$

The unique solution to the combined equations is $\lambda_2 = \frac{1}{6}$, $\lambda_3 = 0$. A single degree of freedom $\lambda_1 \in \mathbb{R}$ remains.

If we only request a form to compute $M_{1,0}$ moments, but not simultaneously $M_{0,1}$, then the choice of the two parameters $\lambda_1,\ \lambda_3 \in \mathbb{R}$ is arbitrary.

Approach 2 We demonstrate the reduction of variables. A solution exists with the symmetry $A^{(1)}_{i_1,i_2,i_3} = A^{(1)}_{i_2,i_1,i_3}$ for all $i_1,\ i_2,\ i_3 \in \{1,\ 2\}$. Specifically, we may identify $A^{(1)}_{2,1,1} = A^{(1)}_{1,2,1}$, and $A^{(1)}_{2,1,2} = A^{(1)}_{1,2,2}$. Then, we are left with the equations

$$(E - I).\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.x' = \frac{1}{8}\begin{pmatrix} 1 & 8 & 2 & 4 & 4 & 1 \\ 1 & -2 & 2 & 0 & 2 & 1 \\ 1 & -2 & 2 & 0 & 2 & 1 \\ 1 & 4 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 & 4 & 1 \\ 1 & 2 & 0 & 2 & -2 & 1 \\ 1 & 2 & 0 & 2 & -2 & 1 \\ 1 & 4 & 4 & 2 & 8 & 1 \end{pmatrix}.x' = 0$$

where $x' = \left(A^{(1)}_{1,1,1},\ A^{(1)}_{1,2,1},\ A^{(1)}_{2,2,1},\ A^{(1)}_{1,1,2},\ A^{(1)}_{1,2,2},\ A^{(1)}_{2,2,2}\right)$. The nullspace is 2-dimensional. The forms are spanned by $\lambda_1,\ \lambda_2 \in \mathbb{R}$ as

$$A^{(1)}_{..,1} = \begin{pmatrix} -\lambda_1 & -\lambda_2 \\ -\lambda_2 & -2\,\lambda_2 \end{pmatrix},\ A^{(1)}_{..,2} = \begin{pmatrix} 2\,\lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}.$$

For calibration, we equate the form applied to P_c to the analytical result from Exercise 1 as

$$M_{1,0}(P_c) = 3\,\lambda_2 = \frac{1}{1+1}\,\frac{1}{1} = \frac{1}{2},\text{ and } M_{0,1}(P_c) = 3\,\lambda_2 = \frac{1}{1}\,\frac{1}{1+1} = \frac{1}{2}.$$

The solution is $\lambda_2 = \frac{1}{6}$. A single degree of freedom remains that we reparameterize to $\lambda_1 = \frac{1}{6}\,\lambda$.

$$A^{(1)}_{..,1} = \frac{1}{6}\begin{pmatrix} -\lambda & -1 \\ -1 & -2 \end{pmatrix},\ A^{(1)}_{..,2} = \frac{1}{6}\begin{pmatrix} 2 & 1 \\ 1 & \lambda \end{pmatrix}.$$

($d = 2$): The 4-form $A^{(2)}$ that computes the moments of degree 2 is

$$A^{(2)}_{..,1,1} = \frac{1}{12}\begin{pmatrix} -\lambda & -1 \\ -1 & -1 \end{pmatrix},\ A^{(2)}_{..,2,1} = \frac{1}{12}\begin{pmatrix} -1 & -1 \\ -1 & -3 \end{pmatrix},\ A^{(2)}_{..,1,2} = \frac{1}{12}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix},\ A^{(2)}_{..,2,2} = \frac{1}{12}\begin{pmatrix} 1 & 1 \\ 1 & \lambda \end{pmatrix}.$$

($d = 3$): The 5-form $A^{(3)}$ that determines the moments of degree 3 is

$$A^{(3)}_{..,1,1,1} = \frac{1}{60}\begin{pmatrix} -\lambda & -3 \\ -3 & -2 \end{pmatrix},\ A^{(3)}_{..,2,1,1} = \frac{1}{60}\begin{pmatrix} -3 & -2 \\ -2 & -3 \end{pmatrix},\ A^{(3)}_{..,1,2,1} = \frac{1}{60}\begin{pmatrix} -3 & -2 \\ -2 & -3 \end{pmatrix},\ A^{(3)}_{..,2,2,1} = \frac{1}{60}\begin{pmatrix} -2 & -3 \\ -3 & -12 \end{pmatrix},$$

$$A^{(3)}_{..,1,1,2} = \frac{1}{60}\begin{pmatrix} 12 & 3 \\ 3 & 2 \end{pmatrix},\ A^{(3)}_{..,2,1,2} = \frac{1}{60}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix},\ A^{(3)}_{..,1,2,2} = \frac{1}{60}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix},\ A^{(3)}_{..,2,2,2} = \frac{1}{60}\begin{pmatrix} 2 & 3 \\ 3 & \lambda \end{pmatrix}.$$

The degree of freedom $\lambda \in \mathbb{R}$ in the calibrated multilinear form $A^{(d)}$ is in the first, and last entry subject to $-A^{(d)}_{1,\ldots,1,1} = A^{(d)}_{2,\ldots,2,2} = \lambda$. The integral solution $\overline{A}^{(d)}$ is easily identified, for instance via

$$\overline{A}^{(d)}_{2,\ldots,2,2} = \int_D b_2(t)^{d+1}\,(\partial_t b_2(t))\,dt = \int_0^1 t^{d+1}\,dt = \tfrac{1}{d+2} \qquad \text{for } d = 0, 1, 2, \ldots$$

Quadratic B-Spline Scheme

The quadratic B-spline scheme is identical to Chaikin's scheme that served as an example throughout the introduction. Here, we only state the calibrated forms $A^{(d)}$. The single degree of freedom is denoted by parameter $\lambda \in \mathbb{R}$.

($d = 0$):

$$A^{(0)} = \tfrac{1}{24}\begin{pmatrix} 0 & -5 & -1 \\ 5 & 0 & -5 \\ 1 & 5 & 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

($d = 1$): We decompose the trilinear form into $A^{(1)} = Y + \lambda\, X$ with

$$Y^{(1)}_{..,1} = \tfrac{1}{240}\begin{pmatrix} 0 & -9 & -1 \\ -9 & -44 & -7 \\ -1 & -7 & -2 \end{pmatrix},\ Y^{(1)}_{..,2} = \tfrac{1}{240}\begin{pmatrix} 18 & 22 & 0 \\ 22 & 0 & -22 \\ 0 & -22 & -18 \end{pmatrix},\ Y^{(1)}_{..,3} = \tfrac{1}{240}\begin{pmatrix} 2 & 7 & 1 \\ 7 & 44 & 9 \\ 1 & 9 & 0 \end{pmatrix},$$

and

$$X^{(1)}_{..,1} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X^{(1)}_{..,2} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ X^{(1)}_{..,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

($d = 2$): With $F = 6720$ as denominator, the coefficients of $A^{(2)} = Y + \lambda\, X$ simplify to

$$Y^{(2)}_{..,1,1} = \tfrac{1}{F}\begin{pmatrix} 0 & -65 & -5 \\ -65 & -237 & -20 \\ -5 & -20 & -3 \end{pmatrix},\ Y^{(2)}_{..,2,1} = \tfrac{1}{F}\begin{pmatrix} -65 & -237 & -20 \\ -237 & -927 & -138 \\ -20 & -138 & -38 \end{pmatrix},\ Y^{(2)}_{..,3,1} = \tfrac{1}{F}\begin{pmatrix} -5 & -20 & -3 \\ -20 & -138 & -38 \\ -3 & -38 & -15 \end{pmatrix},$$

$$Y^{(2)}_{..,1,2} = \tfrac{1}{F}\begin{pmatrix} 195 & 237 & 2 \\ 237 & 309 & 0 \\ 2 & 0 & -2 \end{pmatrix},\ Y^{(2)}_{..,2,2} = \tfrac{1}{F}\begin{pmatrix} 237 & 309 & 0 \\ 309 & 0 & -309 \\ 0 & -309 & -237 \end{pmatrix},\ Y^{(2)}_{..,3,2} = \tfrac{1}{F}\begin{pmatrix} 2 & 0 & -2 \\ 0 & -309 & -237 \\ -2 & -237 & -195 \end{pmatrix},$$

$$Y^{(2)}_{..,1,3} = \tfrac{1}{F}\begin{pmatrix} 15 & 38 & 3 \\ 38 & 138 & 20 \\ 3 & 20 & 5 \end{pmatrix},\ Y^{(2)}_{..,2,3} = \tfrac{1}{F}\begin{pmatrix} 38 & 138 & 20 \\ 138 & 927 & 237 \\ 20 & 237 & 65 \end{pmatrix},\ Y^{(2)}_{..,3,3} = \tfrac{1}{F}\begin{pmatrix} 3 & 20 & 5 \\ 20 & 237 & 65 \\ 5 & 65 & 0 \end{pmatrix},$$

and

$$X^{(2)}_{..,1,1} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X^{(2)}_{..,2,1} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X^{(2)}_{..,3,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X^{(2)}_{..,1,2} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X^{(2)}_{..,2,2} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ X^{(2)}_{..,3,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$X^{(2)}_{..,1,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X^{(2)}_{..,2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ X^{(2)}_{..,3,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The integral solution $\overline{A}^{(d)}$ is easily identified, for instance via

$$\overline{A}^{(d)}_{3,\ldots,3,3} = \int_D b_3(t)^{d+1}\,(\partial_t b_3(t))\,dt = \int_D \left(\tfrac{1}{2}\,t^2\right)^{d+1} t\,dt = \tfrac{1}{d+2}\,2^{-(d+2)} \qquad \text{for } d = 0, 1, 2, \ldots$$

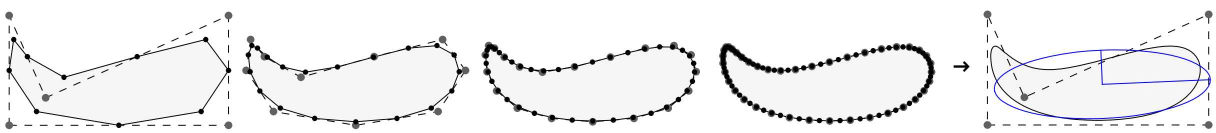
Cubic B-Spline Scheme

A popular scheme is cubic B-spline subdivision. It uses the following averaging mask, and mid-edge insertion

A single refinement step $S(P)$ introduces two points with coordinates

$$\left(\frac{3}{4}\,\mathrm{px}_k + \frac{1}{8}\,(\mathrm{px}_{k-1} + \mathrm{px}_{k+1}),\ \frac{3}{4}\,\mathrm{py}_k + \frac{1}{8}\,(\mathrm{py}_{k-1} + \mathrm{py}_{k+1})\right), \text{ and } \left(\frac{1}{2}\,(\mathrm{px}_k + \mathrm{px}_{k+1}),\ \frac{1}{2}\,(\mathrm{py}_k + \mathrm{py}_{k+1})\right)$$

for all $k = 1, 2, \ldots, n$. The old points $(\mathrm{px}_k, \mathrm{py}_k) \in P$ are dropped.



Example: Several iterations of cubic B-spline subdivision. The moments are $\mathrm{area}(\Omega) = 8/9$, $\mathrm{centroid}(\Omega) = \left(\frac{125639}{120960}, \frac{59257}{161280}\right)$, and $\mathrm{inertia}(\Omega - \mathrm{centroid}(\Omega)) = \left(\frac{3719211599}{16460236800}, \frac{2119004507}{241416806400}, \frac{7388274661}{321889075200}\right)$. ■

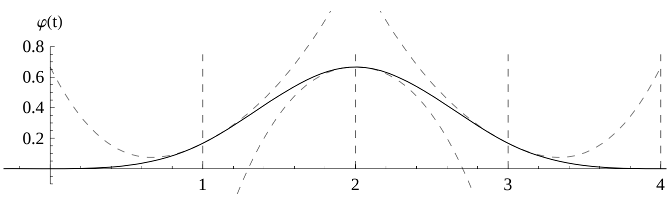


Figure: The basis function segments are the cubic polynomials $b_1(t) = \frac{1}{6}(1-t)^3$, $b_2(t) = \frac{1}{6}\left(4 - 6t^2 + 3t^3\right)$, $b_3(t) = \frac{1}{6}\left(1 + 3t + 3t^2 - 3t^3\right)$, and $b_4(t) = \frac{1}{6}\,t^3$ for $t \in D$. ■

The refinement matrices are

$$S^1 = \frac{1}{8}\begin{pmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{pmatrix}, \text{ and } S^2 = \frac{1}{8}\begin{pmatrix} 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 4 & 4 \end{pmatrix}.$$

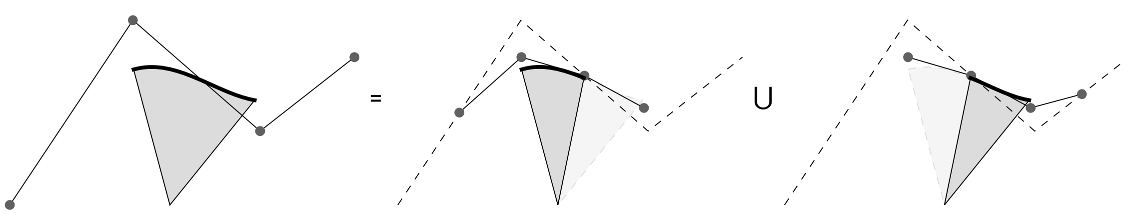


Figure: A facet is determined by $m = 4$ control points. ■

($d = 0$): The bilinear form $A^{(0)}$ that computes the area enclosed by the subdivision curve is

$$A^{(0)} = \frac{1}{720}\begin{pmatrix} 0 & -31 & -28 & -1 \\ 31 & 0 & -183 & -28 \\ 28 & 183 & 0 & -31 \\ 1 & 28 & 31 & 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 & -4 & -1 & 0 \\ -4 & -15 & 0 & 1 \\ -1 & 0 & 15 & 4 \\ 0 & 1 & 4 & 1 \end{pmatrix}.$$

($d = 1$): The trilinear form decomposes as $A^{(1)} = Y + \lambda X$ for $\lambda \in \mathbb{R}$

$$A^{(1)}_{..,1} = \frac{1}{181\,440}\begin{pmatrix} 0 & -485 & -350 & -5 \\ -485 & -6350 & -4229 & -108 \\ -350 & -4229 & -3188 & -129 \\ -5 & -108 & -129 & -10 \end{pmatrix} + \lambda \begin{pmatrix} -1 & -4 & -1 & 0 \\ -4 & -16 & -4 & 0 \\ -1 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{(1)}_{..,2} = \frac{1}{181\,440}\begin{pmatrix} 970 & 3175 & 328 & -21 \\ 3175 & 0 & -14\,181 & -1594 \\ 328 & -14\,181 & -28\,362 & -3901 \\ -21 & -1594 & -3901 & -700 \end{pmatrix} + \lambda \begin{pmatrix} -4 & -16 & -4 & 0 \\ -16 & -63 & -12 & 1 \\ -4 & -12 & 12 & 4 \\ 0 & 1 & 4 & 1 \end{pmatrix}, \ldots$$

The coefficients in $A^{(1)}_{..,3}$, and $A^{(1)}_{..,4}$ follow from the discussed symmetry. For instance, $A^{(1)}_{4,2,3} = -A^{(1)}_{1,3,2} = -\frac{328}{181\,440} + 4\,\lambda$.

($d = 2$): We state only the first few coefficients of $A^{(2)}$

$$A^{(2)}_{..,1,1} = \frac{1}{11\,975\,040}\begin{pmatrix} 0 & -2786 & -1820 & -14 \\ -2786 & -30\,837 & -16\,692 & -175 \\ -1820 & -16\,692 & -9072 & -136 \\ -14 & -175 & -136 & -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 & -4 & -1 & 0 \\ -4 & -16 & -4 & 0 \\ -1 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ldots$$

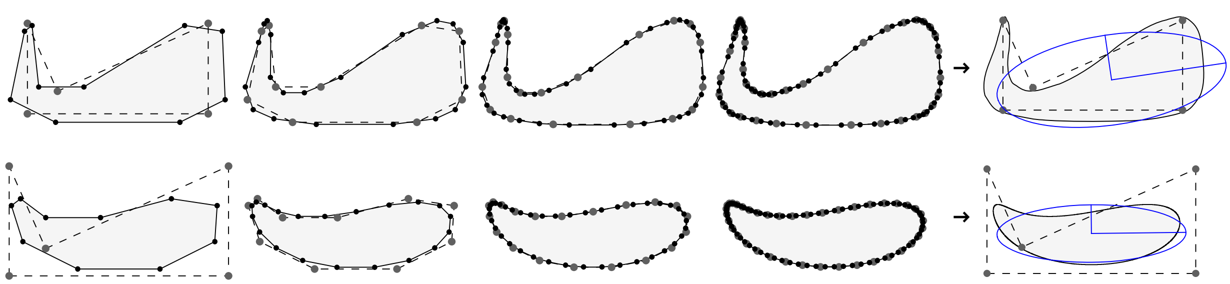
The integral solution $\overline{A}^{(d)}$ is easily identified, for instance via

$$\overline{A}^{(d)}_{4,\ldots,4,4} = \int_D b_4(t)^{d+1}\,(\partial_t b_4(t))\,dt = \int_D \left(\frac{1}{6}\,t^3\right)^{d+1} \left(\frac{1}{2}\,t^2\right) dt = \frac{1}{d+2}\,6^{-(d+2)} \qquad \text{for } d = 0, 1, 2, \ldots$$

Dual Three-Point Scheme

We blend quartic B-spline subdivision with the three-point scheme from [Hormann/Sabin 2008] using a tension parameter $\omega \in \mathbb{R}$. The affine linear combinations are depicted as

[Hormann/Sabin 2008] derive the subdivision rules for the case $\omega = 1/32$ as follows: "*the two new points adjacent to a given old point are taken by sampling a quadratic through three adjacent old points. It therefore has quadratic precision by construction.*" For $\omega = -1/48$, the weights match the quartic B-spline refinement.



Example: Several iterations using the dual three-point scheme for $\omega = 1/32$, and $\omega = -1/48$ below. In the first case, the moments are $\text{area}(\Omega) = \frac{1\,456\,445}{850\,944}$, and $\text{centroid}(\Omega) = \left(\frac{44\,666\,484\,485\,295\,089}{36\,926\,311\,571\,244\,900}, \frac{33\,257\,983\,105\,353\,359}{98\,470\,164\,189\,986\,400}\right)$. With $\Omega_0 = \Omega - \text{centroid}(\Omega)$ the $\text{inertia}(\Omega_0)$ is defined by

$$M_{2,0}(\Omega_0) = 32\,137\,042\,929\,772\,663\,012\,437\,962\,049\,279\,564\,628\,673\,087\,849\,333\,915\,076\,973\,555\,465\,457 / 38\,943\,309\,347\,110\,449\,308\,760\,420\,808\,529\,781\,447\,226\,085\,360\,717\,752\,602\,094\,141\,440\,000,$$

$$M_{1,1}(\Omega_0) = 2\,689\,489\,081\,140\,368\,354\,080\,449\,370\,909\,309\,347\,338\,305\,597\,043\,084\,628\,297\,632\,283\,557 / 25\,962\,206\,231\,406\,966\,205\,840\,280\,539\,019\,854\,298\,150\,723\,573\,811\,835\,068\,062\,760\,960\,000,$$

$$M_{0,2}(\Omega_0) = 9\,892\,097\,824\,294\,533\,258\,996\,926\,970\,294\,442\,434\,640\,171\,474\,427\,755\,082\,099\,646\,944\,759 / 69\,232\,549\,950\,418\,576\,548\,907\,414\,770\,719\,611\,461\,735\,262\,863\,498\,226\,848\,167\,362\,560\,000\,. \blacksquare$$

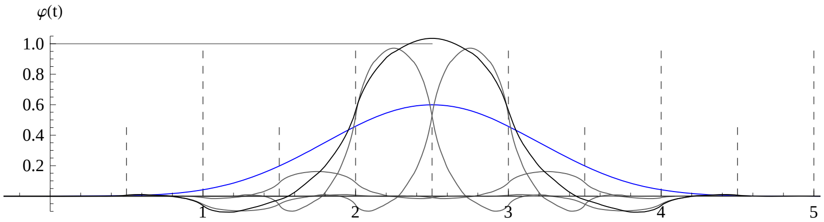


Figure: The basis function φ has support in $[0, 5] \subset \mathbb{R}$ and generally does not have a closed-form expression. The refinement is shown for $\omega = 1/32$. The quartic B-spline is in blue for comparison with the basis function segments as $b_1(t) = \frac{1}{24}(1-t)^4$, $b_2(t) = \frac{1}{24}(11 - 12t - 6t^2 + 12t^3 - 4t^4)$, $b_3(t) = \frac{1}{24}(11 + 12t - 6t^2 - 12t^3 + 6t^4)$, $b_4(t) = \frac{1}{24}(1 + 4t + 6t^2 + 4t^3 - 4t^4)$, and $b_5(t) = \frac{1}{24}t^4$ for $t \in [0, 1]$. ■

The matrices S^h that map the control points of facet f to the control points of facets f_h for $h \in \{1, 2\}$ are

$$S^1 = \begin{pmatrix} \frac{1}{4} - 3\omega & \frac{3}{4} + 6\omega & -3\omega & 0 & 0 \\ -3\omega & \frac{3}{4} + 6\omega & \frac{1}{4} - 3\omega & 0 & 0 \\ 0 & \frac{1}{4} - 3\omega & \frac{3}{4} + 6\omega & -3\omega & 0 \\ 0 & -3\omega & \frac{3}{4} + 6\omega & \frac{1}{4} - 3\omega & 0 \\ 0 & 0 & \frac{1}{4} - 3\omega & \frac{3}{4} + 6\omega & -3\omega \end{pmatrix}, S^2 = \begin{pmatrix} -3\omega & \frac{3}{4} + 6\omega & \frac{1}{4} - 3\omega & 0 & 0 \\ 0 & \frac{1}{4} - 3\omega & \frac{3}{4} + 6\omega & -3\omega & 0 \\ 0 & -3\omega & \frac{3}{4} + 6\omega & \frac{1}{4} - 3\omega & 0 \\ 0 & 0 & \frac{1}{4} - 3\omega & \frac{3}{4} + 6\omega & -3\omega \\ 0 & 0 & -3\omega & \frac{3}{4} + 6\omega & \frac{1}{4} - 3\omega \end{pmatrix}.$$

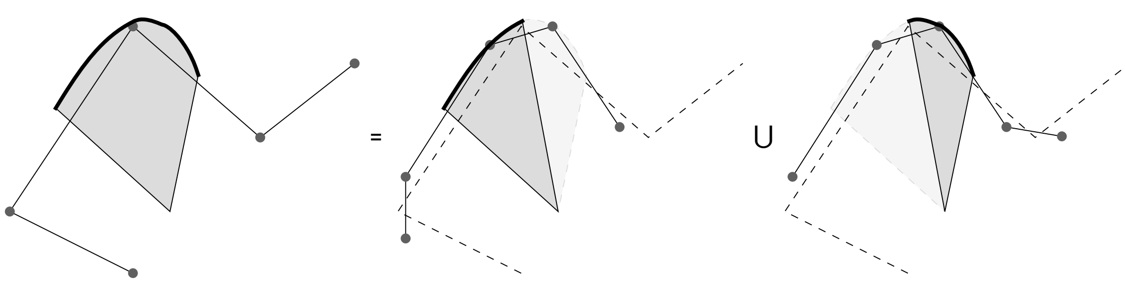


Figure: We subdivide a facet f into f_1 and f_2 with $\omega = 1/32$. Each facet is determined by $m = 5$ control points. ■

($d = 0$): The bilinear form $A^{(0)}$ that computes the area enclosed by the subdivision curve is

$$A^{(0)} = \frac{1}{F}\begin{pmatrix} 0 & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ -a_{1,2} & 0 & a_{2,3} & a_{2,4} & a_{1,4} \\ -a_{1,3} & -a_{2,3} & 0 & a_{2,3} & a_{1,3} \\ -a_{1,4} & -a_{2,4} & -a_{2,3} & 0 & a_{1,2} \\ -a_{1,5} & -a_{1,4} & -a_{1,3} & -a_{1,2} & 0 \end{pmatrix} + \lambda \begin{pmatrix} -16\omega^2 & \mu_1 & \mu_1 & -16\omega^2 & 0 \\ \mu_1 & -1 - 8\omega & -\mu_2 & 0 & 16\omega^2 \\ \mu_1 & -\mu_2 & 0 & \mu_2 & -\mu_1 \\ -16\omega^2 & 0 & \mu_2 & 1 + 8\omega & -\mu_1 \\ 0 & 16\omega^2 & -\mu_1 & -\mu_1 & 16\omega^2 \end{pmatrix}.$$

for $\lambda \in \mathbb{R}$, $F = 120(14 - 27\omega + 432\omega^2 - 1728\omega^3)$, and $a_{1,2} = 16\omega(26 - 21\omega + 2799\omega^2 + 11\,664\omega^3 + 36\,288\omega^4)$, $a_{1,3} = -4\omega(-302 - 1653\omega + 2700\omega^2 + 110\,592\omega^3 + 145\,152\omega^4)$, $a_{1,4} = -4\omega(-7 + 24\omega)(2 - 333\omega - 540\omega^2 + 6048\omega^3)$, $a_{1,5} = 144\omega^3(-7 + 24\omega)(-1 + 168\omega)$, $a_{2,3} = -350 - 6181\omega - 45\,084\omega^2 - 116\,640\omega^3 + 366\,336\omega^4 + 1\,161\,216\omega^5$, $a_{2,4} = -70 + 3991\omega + 47\,784\omega^2 + 92\,736\omega^3 + 13\,824\omega^4$, and $\mu_1 = 4\omega(1 + 4\omega)$, $\mu_2 = 1 + 12\omega + 32\omega^2$.

The entries independent of $\lambda \in \mathbb{R}$ correspond to the otherwise unknown integral solution, for instance

$$\overline{A}^{(0)}_{5,1} = A^{(0)}_{5,1} = \frac{6\omega^3(-1+168\omega)}{5(2+3\omega+72\omega^2)}.$$

Example: The area defined by subdivision of $P = ((0, 0), (1, 0), (1, 1), (0, 1))$ is

$$M_{0,0}(P) = \frac{50-1003\,\omega+8568\,\omega^2-42\,048\,\omega^3+96\,768\,\omega^4}{60-90\,\omega+2160\,\omega^2}. \blacksquare$$

($\boldsymbol{d}$ = 1): We compute the trilinear form $A^{(1)}$ for the moments of degree 1 with variable $\omega \in \mathbb{R}$. The denominators of the rational coefficients have the least common multiple

$$20\,160\,\omega^3\,(3+8\,\omega)\,(-31+24\,\omega)\,(-7+24\,\omega)\left(2+3\,\omega+72\,\omega^2\right)\left(8+3\,\omega+72\,\omega^2\right)$$
$$\left(16+3\,\omega+144\,\omega^2+1728\,\omega^3\right)\left(-124+99\,\omega+1368\,\omega^2+5184\,\omega^3+41\,472\,\omega^4\right)$$

Because the terms are lengthy, we state the tensor here only for the special case $\omega = 1/32$. Then, with $F = 462\,775\,243\,294\,780\,416\,000$, the values are

$$A^{(1)}_{..,1} = \frac{1}{F}\begin{pmatrix} -\lambda & -75\,616\,865\,897\,512\,347+9\,\lambda & -260\,673\,393\,176\,499\,501+9\,\lambda & 35\,129\,695\,124\,510\,163-\lambda & -125\,401\,737\,204\,315 \\ * & 1\,973\,889\,859\,785\,860\,592-81\,\lambda & 5\,578\,664\,264\,397\,201\,582-81\,\lambda & -707\,774\,131\,993\,302\,432+9\,\lambda & 873\,920\,433\,665\,565 \\ * & 5\,578\,664\,264\,397\,201\,582-81\,\lambda & 11\,627\,814\,980\,844\,902\,484-81\,\lambda & -1\,814\,434\,036\,684\,694\,190+9\,\lambda & -34\,356\,268\,056\,960\,135 \\ * & -707\,774\,131\,993\,302\,432+9\,\lambda & -1\,814\,434\,036\,684\,694\,190+9\,\lambda & 236\,320\,804\,246\,939\,344-\lambda & 575\,698\,863\,740\,715 \\ * & 873\,920\,433\,665\,565 & -34\,356\,268\,056\,960\,135 & 575\,698\,863\,740\,715 & -250\,803\,474\,408\,630 \end{pmatrix},$$

$$A^{(1)}_{..,2} = \frac{1}{F}\begin{pmatrix} 151\,233\,731\,795\,024\,694+9\,\lambda & * & -495\,234\,794\,685\,929\,220-81\,\lambda & -15\,645\,450\,011\,445\,288+9\,\lambda & -298\,221\,569\,924\,850 \\ -986\,944\,929\,892\,930\,296-81\,\lambda & 730\,\lambda & -25\,811\,094\,537\,690\,099\,120+720\,\lambda & 2\,577\,001\,810\,523\,079\,744-90\,\lambda & 118\,160\,402\,123\,469\,672+\lambda \\ -495\,234\,794\,685\,929\,220-81\,\lambda & * & -190\,750\,492\,268\,067\,767\,688+810\,\lambda & 22\,725\,776\,073\,818\,600\,496 & -1\,112\,312\,027\,867\,211\,828-9\,\lambda \\ -15\,645\,450\,011\,445\,288+9\,\lambda & * & 22\,725\,776\,073\,818\,600\,496 & 5\,154\,003\,621\,046\,159\,488+90\,\lambda & -723\,419\,582\,004\,747\,720-9\,\lambda \\ -298\,221\,569\,924\,850 & * & -1\,112\,312\,027\,867\,211\,828-9\,\lambda & -723\,419\,582\,004\,747\,720-9\,\lambda & 70\,259\,390\,249\,020\,326+\lambda \end{pmatrix},$$

$$A^{(1)}_{..,3} = \frac{1}{F}\begin{pmatrix} 521\,346\,786\,352\,999\,002+9\,\lambda & -5\,083\,429\,469\,711\,272\,362-81\,\lambda & * & 702\,122\,008\,817\,482\,362+9\,\lambda & 0 \\ -5\,083\,429\,469\,711\,272\,362-81\,\lambda & 51\,622\,189\,075\,380\,198\,240+720\,\lambda & * & 0 & -702\,122\,008\,817\,482\,362-9\,\lambda \\ -5\,813\,907\,490\,422\,451\,242-81\,\lambda & 95\,375\,246\,134\,033\,883\,844+810\,\lambda & 0 & -95\,375\,246\,134\,033\,883\,844-810\,\lambda & 5\,813\,907\,490\,422\,451\,242+81\,\lambda \\ 702\,122\,008\,817\,482\,362+9\,\lambda & 0 & * & -51\,622\,189\,075\,380\,198\,240-720\,\lambda & 5\,083\,429\,469\,711\,272\,362+81\,\lambda \\ 0 & -702\,122\,008\,817\,482\,362-9\,\lambda & * & 5\,083\,429\,469\,711\,272\,362+81\,\lambda & -521\,346\,786\,352\,999\,002-9\,\lambda \end{pmatrix}.$$

The coefficients in $A^{(1)}_{..,4}$, and $A^{(1)}_{..,5}$, as well as the entries supressed by “$*$” follow from the symmetry.

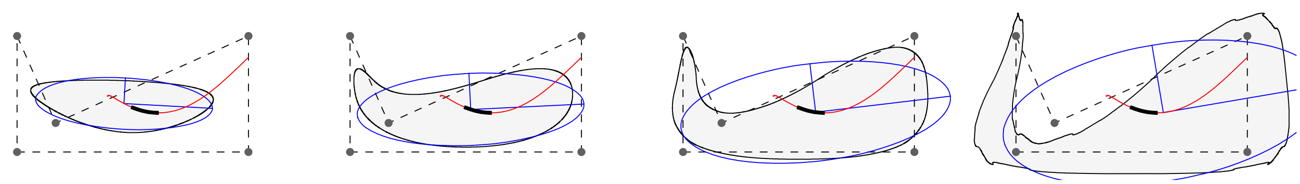
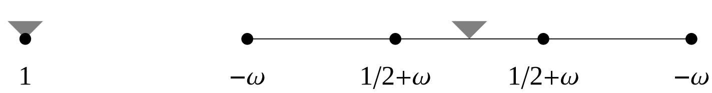


Figure: We trace the centroid for tension parameters in the range $-1/8 < \omega < 1/7$. The thick segment corresponds to the valid range $\omega \in [-1/48,\ 1/32]$. The exhibits are for $\omega \in \{-1/24,\ -1/96,\ 1/64,\ 3/64\}$. ■

(2 ≤ $\boldsymbol{d}$): Our attempts to obtain $A^{(2)}$ for general $\omega \in \mathbb{R}$ were not successful. For any specific tension parameter however, we compute the multilinear forms for moments up to degree 3 without difficulty.

Interpolatory C^1 Four-Point Scheme

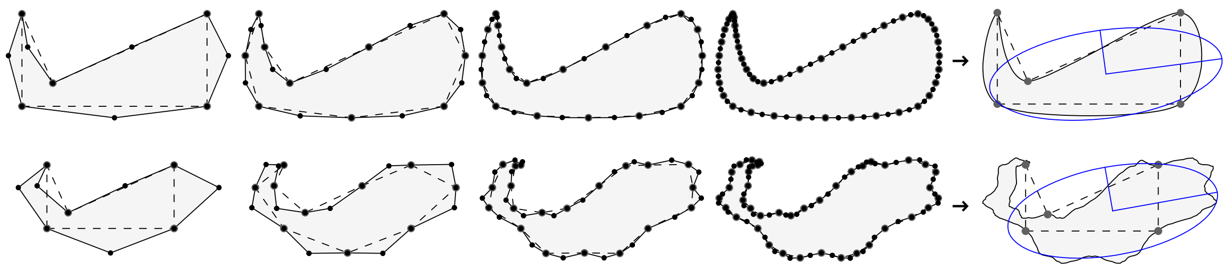
[Dubuc 1986] designed the interpolatory four-point scheme to mimic cubic polynomial reproduction in each step. [Dyn/Gregory/Levin 1987] introduced a tension parameter $\omega \in \mathbb{R}$ to blend the curves with linear B-spline subdivision. This has made a lot of people very angry and been widely regarded as a bad move. The weights are



The scheme is interpolatory, that means $(\mathrm{px}_k,\ \mathrm{py}_k) \in P$ are preserved by $S(P)$. Additionally, a point with coordinates $\left(\left(\frac{1}{2}+\omega\right)(\mathrm{px}_k+\mathrm{px}_{k+1}) - \omega(\mathrm{px}_{k-1}+\mathrm{px}_{k+2}),\ \left(\frac{1}{2}+\omega\right)(\mathrm{py}_k+\mathrm{py}_{k+1}) - \omega(\mathrm{py}_{k-1}+\mathrm{py}_{k+2})\right)$ is introduced for all $k = 1,\ 2,\ \ldots,\ n$.

[Hechler/Moessner/Reif 2008] prove that the basis function φ is C^1 for exactly $\omega \in (0,\ \omega^*) \subset \mathbb{R}$ where

$$\omega^* = \frac{1}{12}\sqrt[3]{27+3\sqrt{105}} - \frac{1}{2}\sqrt[-3]{27+3\sqrt{105}} = 0.19272924926481202520628659232675674181376\ldots$$



Example: Subdivision with tension parameters $\omega = 1/16$, and $\omega = \omega^*$ below. For $\omega = 1/16$, the moments evaluate as $\text{area}(\Omega) = \frac{446\,389}{266\,112}$, and $\text{centroid}(\Omega) = \left(\frac{7\,692\,606\,932\,638\,356\,977}{6\,491\,763\,064\,547\,046\,864}, \frac{5\,697\,393\,899\,777\,829\,797}{17\,311\,368\,172\,125\,458\,304}\right)$. ■

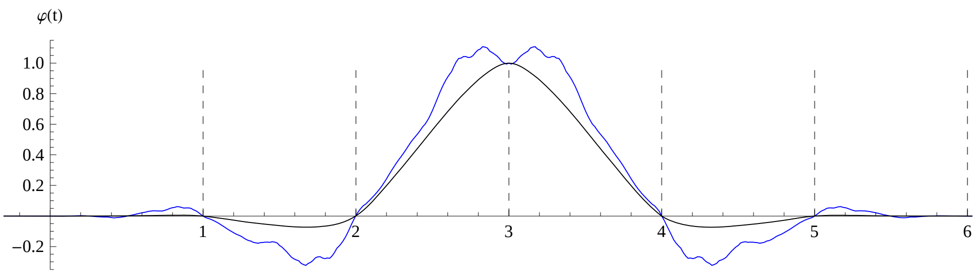


Figure: The basis function φ has support in the interval $[0, 6] \subset \mathbb{R}$ and no closed-form expression. We plot φ for $\omega = 1/16$ (black), and $\omega = \omega^*$ (blue). ■

The matrices S^h that map the control points of facet f to the control points of facet f_h for $h \in \{1, 2\}$ are

$$S^1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega & \mu & \mu & -\omega & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\omega & \mu & \mu & -\omega & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega & \mu & \mu & -\omega \end{pmatrix}, S^2 = \begin{pmatrix} -\omega & \mu & \mu & -\omega & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\omega & \mu & \mu & -\omega & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega & \mu & \mu & -\omega \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{with } \mu = \frac{1}{2} + \omega.$$

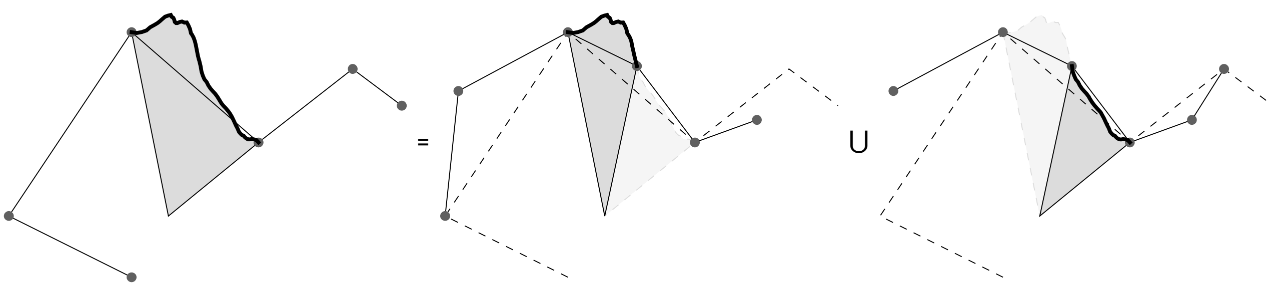


Figure: Decomposition of a facet f into two smaller facets f_1 and f_2 by one round of subdivision with $\omega = \omega^*$. The parameterization of a facet requires $m = 6$ control points. ■

(***d* = 0**)**:** The calibrated bilinear form that determines the area enclosed by the subdivision curves is

$$A^{(0)} = \frac{1}{F} \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ -a_{1,2} & 0 & a_{2,3} & a_{2,4} & a_{2,5} & a_{1,5} \\ -a_{1,3} & -a_{2,3} & -\lambda & a_{3,4} & a_{2,4} & a_{1,4} \\ -a_{1,4} & -a_{2,4} & -a_{3,4} & \lambda & a_{2,3} & a_{1,3} \\ -a_{1,5} & -a_{2,5} & -a_{2,4} & -a_{2,3} & 0 & a_{1,2} \\ -a_{1,6} & -a_{1,5} & -a_{1,4} & -a_{1,3} & -a_{1,2} & 0 \end{pmatrix}$$

for $\lambda \in \mathbb{R}$, $F = 6 - 24\,\omega + 72\,\omega^2 - 102\,\omega^3 + 144\,\omega^4 - 96\,\omega^5$, and the coefficients are $a_{1,2} = 4\,\omega^3 + 4\,\omega^4 + 8\,\omega^5 + 8\,\omega^6$, $a_{1,3} = 2\,\omega^2 - 10\,\omega^3 + 6\,\omega^4 - 16\,\omega^5$, $a_{1,4} = -2\,\omega^2 + 2\,\omega^3 - 6\,\omega^4$, $a_{1,5} = 4\,\omega^3 - 4\,\omega^4$, $a_{1,6} = 8\,\omega^5 - 8\,\omega^6$,

$a_{2,3} = -4\omega + 6\omega^2 - 12\omega^3 + 16\omega^4 - 12\omega^5$, $a_{2,4} = 4\omega - 2\omega^2 + 14\omega^3 - 6\omega^4 + 8\omega^5$, $a_{2,5} = -4\omega^2 - 2\omega^3 - 2\omega^4 + 12\omega^5 + 8\omega^6$, $a_{3,4} = -3 + 4\omega - 24\omega^2 + 13\omega^3 - 38\omega^4 + 12\omega^5$.

We verify the expression stated in [Warren/Weimer 2002] on page 166 for $\omega = 1/16$:

$$\text{area}(\Omega) = \sum_{k=1}^{n} \text{px}_k \left(\frac{3659\,(\text{py}_{k+1} - \text{py}_{k-1})}{5280} - \frac{731\,(\text{py}_{k+2} - \text{py}_{k-2})}{6930} + \frac{481\,(\text{py}_{k+3} - \text{py}_{k-3})}{73\,920} - \frac{4\,(\text{py}_{k+4} - \text{py}_{k-4})}{10\,395} - \frac{\text{py}_{k+5} - \text{py}_{k-5}}{665\,280} \right).$$

Example: The area defined by subdivision of $P = ((0,0), (1,0), (1,1), (0,1))$ is $M_{0,0}(P) = \frac{3 + 7\omega + 11\omega^2 + 16\omega^3}{3 - 9\omega + 27\omega^2 - 24\omega^3 + 48\omega^4}$. For $\omega = 0$, the scheme is linear subdivision, so that the limit curve $S^\infty(P) = \partial([0,1]^2)$ bounds the unit square and the expression for $M_{0,0}(P)$ correctly simplifies to 1. ■

($d = 1$): To establish the nullspace of a matrix that depends on a parameter $\omega \in \mathbb{R}$ is computationally challenging. Making use of the symmetry reduces the number of variables from $6^{1+2} = 216$ to $\binom{1+6}{1+1} 6 = 126$.

The denominators of the rational coefficients in $A^{(1)}$ with variable ω have the least common multiple

$$6(-1+\omega)(14+\omega)(2-\omega+2\omega^3)(1-2\omega+2\omega^2-4\omega^3+8\omega^4)(-28+38\omega-29\omega^2+56\omega^3+8\omega^4)$$
$$(1-3\omega+9\omega^2-8\omega^3+16\omega^4)(-8+18\omega-35\omega^2+22\omega^3+7\omega^4-28\omega^5+112\omega^6-28\omega^7+102\omega^8$$
$$+448\omega^9+48\omega^{10}+24\omega^{11}+128\omega^{12})$$

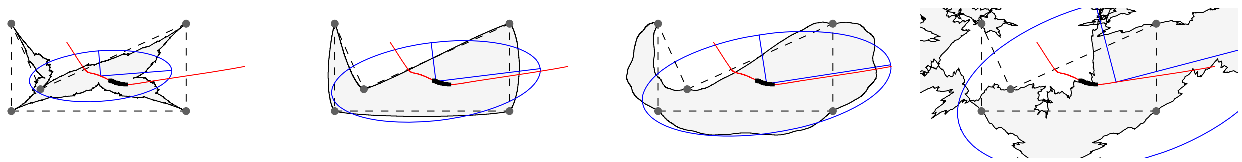


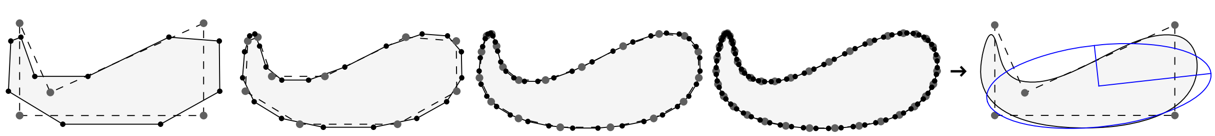
Figure: We trace the centroid for tension parameters in the range $-5/8 < \omega < 0.475$. The thick segment corresponds to the valid range $\omega \in [0, \omega^*)$. The exhibits are for $\omega \in \{-1/6, 1/32, 1/8, 1/3\}$. The last graph is cropped intentionally. ■

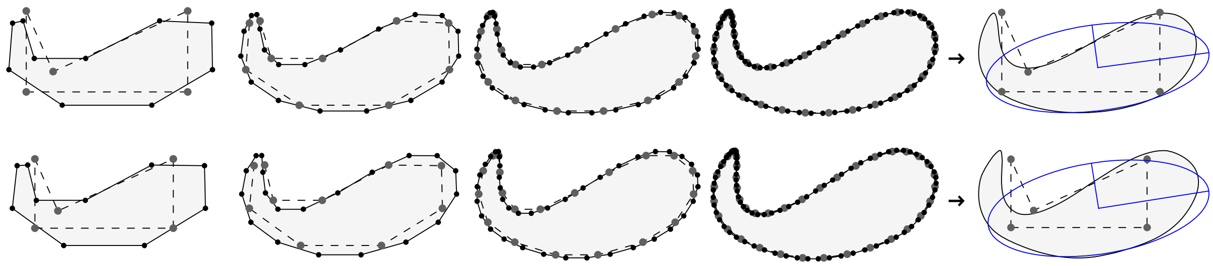
($2 \leq d$): Our attempts to obtain $A^{(2)}$ for general $\omega \in \mathbb{R}$ were not successful. For any specific tension parameter however, we compute the multilinear forms for moments up to degree 3 without difficulty.

Dual C^2 Four-Point Scheme

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened: The dual C^2 four-point scheme was introduced by [Dyn/Floater/Hormann 2005] and uses the tension parameter $\omega \in \mathbb{R}$. The weights are

The default choice is $\omega = 1/128 = 0.0078125$ that mimics cubic polynomial interpolation in every iteration. The parameter ω represents a "*perturbation of Chaikin's scheme*". The authors guarantee C^2 smoothness for parameters in the interval $\omega \in (0, 1/48]$, and do not rule out values beyond $\omega > 1/48 = 0.0208333\ldots$.





Example: Subdivision with tension parameter $\omega = 1/128$, $\omega = \omega^+$, and $\omega = 1/48$.

$$\text{area}(\Omega) = \big(102 + 5223\,\omega + 123\,002\,\omega^2 + 1\,876\,832\,\omega^3 + 19\,818\,880\,\omega^4 + 144\,020\,864\,\omega^5 + 785\,715\,200\,\omega^6 + 3\,008\,593\,920\,\omega^7 + 5\,505\,024\,000\,\omega^8\big) / \big(48\,(2 - 3\,\omega + 94\,\omega^2 + 552\,\omega^3 + 10\,752\,\omega^4 + 40\,960\,\omega^5 + 655\,360\,\omega^6)\big).$$

In case of $\omega = 1/128$, the centroid(Ω) has the coordinates $\frac{450\,509\,098\,442\,668\,672\,336\,597\,625\,038\,810\,289\,839\,103}{389\,642\,527\,795\,896\,514\,871\,877\,005\,174\,261\,012\,835\,780}$, and $\frac{2\,706\,172\,510\,094\,823\,837\,432\,727\,436\,746\,091\,081\,555\,329}{8\,312\,373\,926\,312\,458\,983\,933\,376\,110\,384\,234\,940\,496\,640}$. ■

For the choice $\omega = \omega^+ := 0.013723\ldots$ the scheme is called “tightest” because the corresponding basis function sampled at the locations $\{\varphi(z + 7/2) \approx \delta_{0,z} : z \in \mathbb{Z}\}$ is closest to the Kronecker sequence in the least square sense. The limit curves are almost, but not quite, entirely unlike interpolatory.

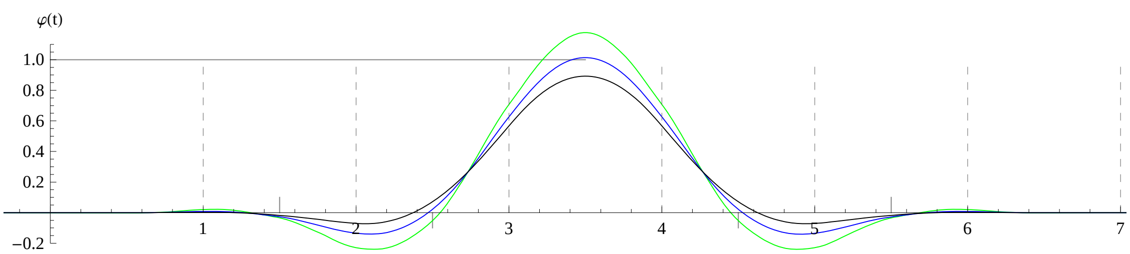


Figure: The basis function φ has support is the interval $[0, 7] \subset \mathbb{R}$ and no closed-form expression. We plot φ for $\omega = 1/128$ (black), $\omega = \omega^+$ (blue), and $\omega = 1/48$ (green). ■

The refinement matrices are

$$S^1 = \begin{pmatrix} -7\,\omega & \mu_1 & \mu_2 & -5\,\omega & 0 & 0 & 0 \\ -5\,\omega & \mu_2 & \mu_1 & -7\,\omega & 0 & 0 & 0 \\ 0 & -7\,\omega & \mu_1 & \mu_2 & -5\,\omega & 0 & 0 \\ 0 & -5\,\omega & \mu_2 & \mu_1 & -7\,\omega & 0 & 0 \\ 0 & 0 & -7\,\omega & \mu_1 & \mu_2 & -5\,\omega & 0 \\ 0 & 0 & -5\,\omega & \mu_2 & \mu_1 & -7\,\omega & 0 \\ 0 & 0 & 0 & -7\,\omega & \mu_1 & \mu_2 & -5\,\omega \end{pmatrix},\; S^2 = \begin{pmatrix} -5\,\omega & \mu_2 & \mu_1 & -7\,\omega & 0 & 0 & 0 \\ 0 & -7\,\omega & \mu_1 & \mu_2 & -5\,\omega & 0 & 0 \\ 0 & -5\,\omega & \mu_2 & \mu_1 & -7\,\omega & 0 & 0 \\ 0 & 0 & -7\,\omega & \mu_1 & \mu_2 & -5\,\omega & 0 \\ 0 & 0 & -5\,\omega & \mu_2 & \mu_1 & -7\,\omega & 0 \\ 0 & 0 & 0 & -7\,\omega & \mu_1 & \mu_2 & -5\,\omega \\ 0 & 0 & 0 & -5\,\omega & \mu_2 & \mu_1 & -7\,\omega \end{pmatrix}.$$

where $\mu_1 = \frac{3}{4} + 9\,\omega$, and $\mu_2 = \frac{1}{4} + 3\,\omega$.

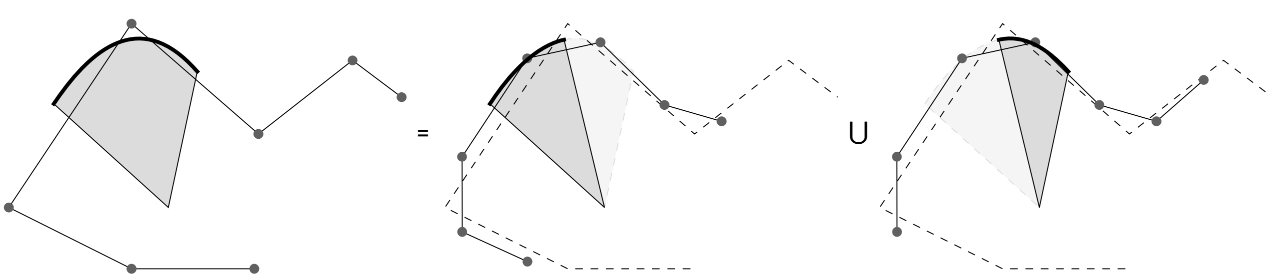
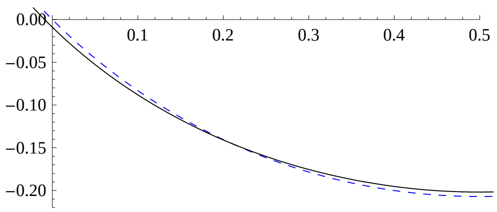


Figure: A facet is determined by $m = 7$ control points. The decomposition is illustrated for $\omega = 1/128$. ■

($d = 0$): The calibrated bilinear form is $A^{(0)} = Y + \lambda X$ where Y is alternating, and X is symmetric.

Example: Subdivision of the simple polygon $P = ((0, 0), (1, 0), (1, 1), (0, 1))$ encloses a set with

$$\text{area}(\Omega) = \left(50 + 2997\,\omega + 67\,118\,\omega^2 + 748\,744\,\omega^3 + 4\,996\,096\,\omega^4 + 19\,599\,872\,\omega^5 + 53\,772\,288\,\omega^6 + 347\,996\,160\,\omega^7 + 880\,803\,840\,\omega^8\right) \Big/ \left(30\left(2 - 3\,\omega + 94\,\omega^2 + 552\,\omega^3 + 10\,752\,\omega^4 + 40\,960\,\omega^5 + 655\,360\,\omega^6\right)\right)$$



The tension parameter $\omega = 0.01122997457488839033860351076962465911\ldots$ produces a curve with area identical to $\pi/2$; the comparison of the limit curve with the circle of radius $1\big/\sqrt{2}$ is shown above. For $\omega = 1/128$, we have $\text{area}(\Omega) = \frac{133\,808\,579\,579}{102\,182\,653\,440}$. For $\omega = 0$, the area simplifies to $5/6$. ■

($d = 1$): The computation of the trilinear form $A^{(1)}$ for general $\omega \in \mathbb{R}$ is facilitated by exploiting the symmetry of the basis function $\varphi(t) = \varphi(7 - t)$ for $t \in \mathbb{R}$, which reduces the number of unknown coefficients to 103. In the resulting tensor $A^{(1)}$, there are 42 entries that depend on ω, but not on the degree of freedom $\lambda \in \mathbb{R}$. The trivial zeros are $A^{(1)}_{i,8-i,4} = \overline{A}^{(1)}_{i,8-i,4} = 0$ for $i = 1, 2, \ldots, 7$.

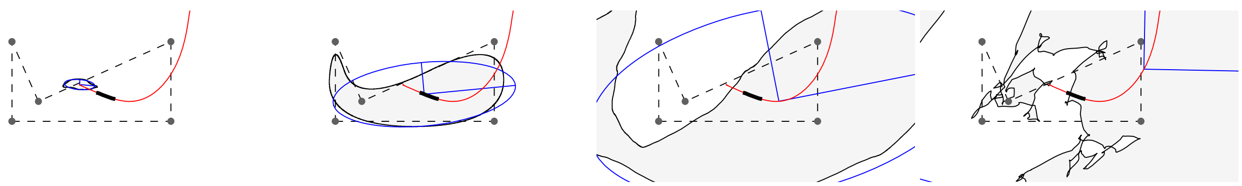
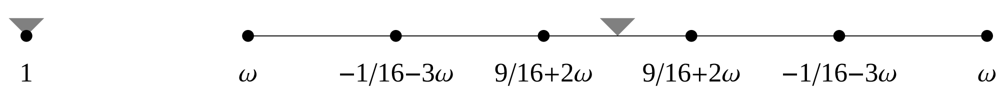


Figure: We trace the centroid for tension parameters in the range $-3/64 < \omega < 1/8$. The thick segment corresponds to the range $\omega \in (0, 1/48]$. The exhibits are for $\omega \in \{-3/64, 1/256, 1/24, 1/12\}$. ■

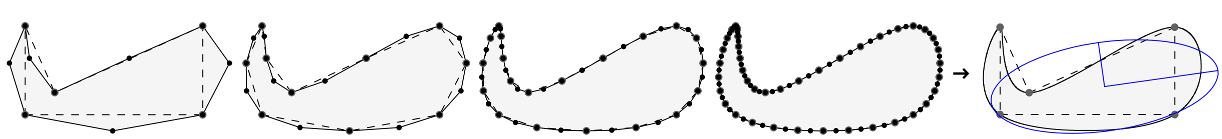
($2 \leq d$): Our attempts to obtain $A^{(2)}$ for general $\omega \in \mathbb{R}$ were not successful. For any specific $\omega \in \mathbb{R}$ however, we obtain the multilinear forms for moments up to degree 2 without difficulty.

Interpolatory C^2 Six-Point Scheme

[Weissman 1990]'s scheme is



The default tension parameter is $\omega = 1/96 = 0.0104\ldots$.



Example: Subdivision with tension parameter $\omega = 1/96$. The limit curve defines $\text{area}(\Omega) = \frac{9\,289\,987\,092\,625\,307}{5\,099\,514\,025\,231\,584}$, and $\text{centroid}(\Omega) = (1.19681406702987269211637191091\ldots, 0.32040951500180181674808897148\ldots)$. ■

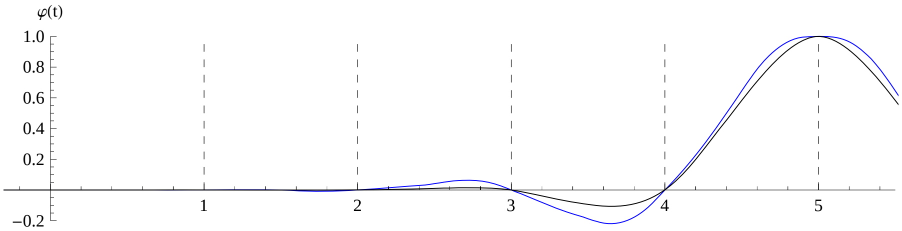


Figure: The basis function φ has support in $[0, 10] \subset \mathbb{R}$ and is shown here for $\omega = 1/96$ (black), and $\omega = 1/24$ (blue). A facet is determined by $m = 10$ control points. ■

($d = 0$): We establish the bilinear form $A^{(0)}$ for general $\omega \in \mathbb{R}$, but do not reproduce the lengthy terms here.

Example: Starting from the unit square $P = ((0, 0), (1, 0), (1, 1), (0, 1))$, the curve encloses an area of

$$M_{0,0}(P) = \left(64\left(-454\,920 - 21\,169\,\omega - 9\,615\,666\,\omega^2 + 14\,971\,600\,\omega^3 + 133\,902\,592\,\omega^4 + 188\,862\,976\,\omega^5 + 965\,378\,048\,\omega^6 + 355\,991\,552\,\omega^7\right)\right) / \left(945\left(-22\,440 + 160\,395\,\omega - 1\,405\,296\,\omega^2 + 4\,406\,704\,\omega^3 - 8\,696\,832\,\omega^4 - 3\,620\,864\,\omega^5 + 104\,857\,600\,\omega^6 - 100\,663\,296\,\omega^7 + 536\,870\,912\,\omega^8\right)\right)$$

For $\omega = 0$, $\text{area}(\Omega) = \frac{14\,272}{10\,395}$ is consistent with the formula for the C^1 four-point scheme stated above. ■

($1 \le d$): Our attempts to obtain $A^{(1)}$ for general $\omega \in \mathbb{R}$ were not successful. For any specific $\omega \in \mathbb{R}$ however, we obtain the multilinear forms for moments up to degree 2 without difficulty.

Final Remarks

The formulas extend to a more general class of curves $S^\infty(P)$ than previously assumed. For instance, if the curves are permitted to self-intersect, then

$$M_{p,q}(P) = \int_{\mathbb{R}^2 \setminus S^\infty(P)} x^p\, y^q\, \nu(x, y)\, dx\, dy$$

where $\nu : \mathbb{R}^2 \setminus S^\infty(P) \to \mathbb{Z}$ gives the winding number of a point in the plane with respect to the curve $S^\infty(P)$.

Let all coefficients in S^h for $h \in \{1, 2\}$ be rational numbers. For an input polygon P with rational coordinates $(\text{px}_k, \text{py}_k) \in P$ for all $k = 1, 2, \ldots, n$, the moment $M_{p,q}(S^\infty(P))$ for $p, q \in \{0, 1, 2, \ldots\}$ is also a rational number.

In the article, we restrict the derivation to binary schemes. Schemes that use k-splits can be handled with the same methodology.

Our choice of the vector field $G_{p,q}$ not only results in a simple derivation, but also leads to a tensor $\frac{1}{p+1} A^{(d)}$ that applies for all $M_{p,q}(\Omega)$ with $p + q = d$ up to the factor $\frac{1}{p+1}$. The more general $\hat{G}_{p,q}(x, y) = \left(\frac{\alpha}{p+1} x^{p+1} y^q, \frac{1-\alpha}{q+1} y^{q+1} x^p\right)$ for fixed $\alpha \in \mathbb{R}$ with $\operatorname{div} \hat{G}_{p,q} = x^p\, y^q$ results in the multilinear form $\hat{A}^{(p,q)}$ that is obtained from $A^{(d)}$ by permutation and averaging

$$\hat{A}^{(p,q)} = \frac{\alpha}{p+1} A^{(d)} - \frac{1-\alpha}{q+1} \tilde{A}^{(d)}$$

where $\tilde{A}^{(d)}$ is the $(d + 2)$-form that has all coefficients of $A^{(d)}$ with indices reversed as

$$\tilde{A}^{(d)}_{i_1,i_2,\ldots,i_{d+1},i_{d+2}} := A^{(d)}_{i_{d+2},i_{d+1},\ldots,i_2,i_1} = A^{(d)}_{i_2,\ldots,i_{d+1},i_{d+2},i_1} \qquad \text{for all } i_1, i_2, \ldots, i_{d+1}, i_{d+2} \in \{1, 2, \ldots, m\}.$$

Then, $M_{p,q}(f) = \sum^m_{i_1,\ldots,i_{p+1},j_1,\ldots,j_{q+1}} \hat{A}^{(p,q)}_{i_1,\ldots,i_{p+1},j_1,\ldots,j_{q+1}}\, \text{px}_{i_1} \ldots \text{px}_{i_{p+1}}\, \text{py}_{j_1} \ldots \text{py}_{j_{q+1}}$.

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