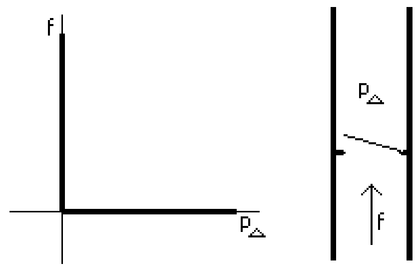

Figure 1 : phase diagram and depicted scenario of airflow through chimney with flap

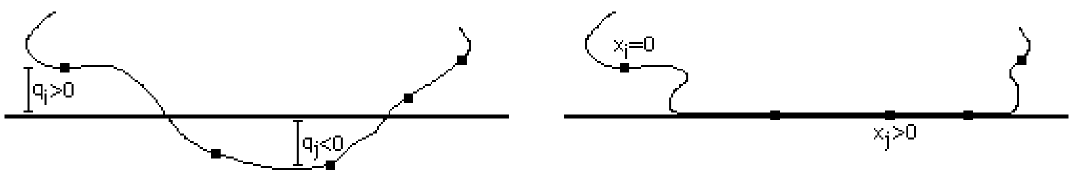

Figure 2 : elastic body colliding with flat surface

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0.1 Linear Complementary Problem

Example 1 (Airflow through chimney). Consider the scenario in figure 1, we denote

$$\begin{aligned} p_\triangle &= \text{pressure excess above door} \\ f &= \text{amount of air flow upwards} \end{aligned}$$

Under the assumption that negative pressure excess is instantaneously compensated by upwards air flow, we yield the equations $f, p_\triangle \geq 0$ and $f p_\triangle = 0$.

Example 2 (Contact problem). Consider an "elastic" body in motion. We want to model the deformation of the body resulting from contact with a flat surface at a certain time, compare to figure 2. We fix a canonic coordinate system and choose n points on the exterior of the body and denote

$$\begin{aligned} x_i &= \text{contact stress at } i\text{-th point} \\ a_{ij} &= \text{rate of excursion due to } j\text{-th stress at } i\text{-th point (usually} > 0 \text{ when } i = j) \\ q_i &= \text{distance of the } i\text{-th point from surface if penetration were permitted (negative when intruding)} \end{aligned}$$

for $i, j = 1...n$. The (unknown) stress vector $x \in \mathbb{R}^n$ is non-negative. Also, $q + Ax \geq 0$, i.e. stresses resolve penetration. The complementary condition is written as

$$< x, q + Ax > \; = x^T(q + Ax) = 0.$$

The scalar product implies that either
- the i-th stress is zero $x_i = 0$ (i-th point does not "try" to penetrate surface) or
- the i-th point lies on surface, i.e. distance $q_i + (Ax)_i = 0$, and $x_i \geq 0$.

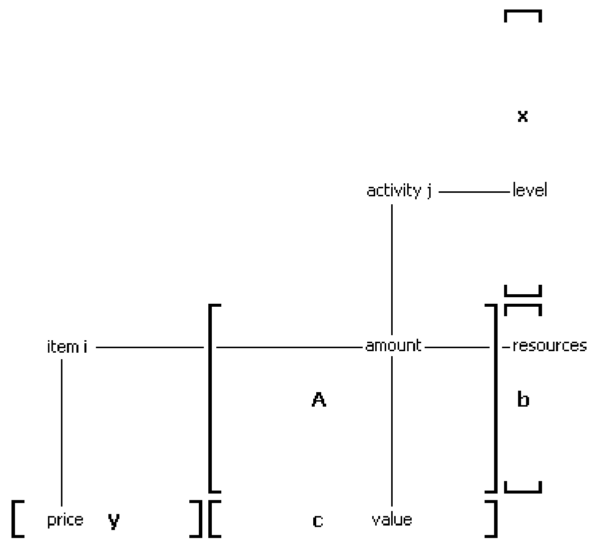

Figure 3 : economical sector with production

Definition 1. A *(linear) complementary problem* is of the form:
Find $x \in \mathbb{R}^n$ with $0 \leq x \perp F(x) \geq 0$ and $F\colon \mathbb{R}^n \to \mathbb{R}^n$ (affine).
"$x \perp F(x)$" means $<x, F(x)> \; = x^T F(x) = 0$.

The solution space for an LCP is invariant under the weakening of the requirement $x^T F(x) = 0$ to $x^T F(x) \leq 0$ as preferred in [AG04]. Suppose $x^T F(x) < 0$, then clearly for some i we must have $x_i F(x)_i < 0$. Let $x_i > 0$ and $F(x)_i < 0$, the latter being a contradiction to $F(x) \geq 0$. Analogous for $x_i < 0$.
An LCP is not an optimization problem. In many applications $F(x)$ is affine, which are multivariate polynomials. In those cases we are dealing with semi-algebraic equations. The next examples show how optimization problems are translated to complementary problems. The procedure is similar to find local maxima or minima of a differentiable $f\colon \mathbb{R} \to \mathbb{R}$ via solving $f'(x) = 0$.

Remark 1 (Linear programming). Let all vectors and matrices be of appropriate dimensions. The dual problem to:
Find $x \geq 0$ with $\max c^T x$ where $b - Ax \geq 0$, is:
Find $y \geq 0$ with $\min y^T b$ where $A^T y - c \geq 0$.
We add slack variables $v, u \geq 0$ into the constraints to yield $b - Ax =: v \geq 0$ and $A^T y - c =: u \geq 0$. One verifies that

$$0 \leq \begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} u \\ v \end{pmatrix} = \left(\begin{array}{c|c} 0 & A^T \\ \hline -A & 0 \end{array}\right) \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -c \\ b \end{pmatrix} \geq 0 \tag{0.1.1}$$

defines the associated LCP, using duality theorems, namely that $x_i > 0$ implies $u_i = 0$ and $y_j > 0$ implies $v_j = 0$.
The Nash equilibrum point of a bimatrix game is the solution to an LP. Another example is to find the best approximation in the 1-norm (or ∞-norm) of a vector by a linear combination of other vectors. Also consider

Example 3 (Equilibrum of economical sector with production). Suppose that an economy has

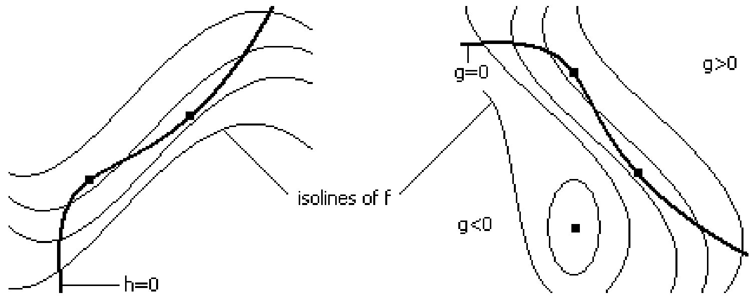

Figure 4 : visualization of lagrange multiplier method, squares indicate local extrema

$i = 1...n$ items available for production and $j = 1...m$ activities (productive processes).

(a_{ij})	$=$	amount of item i required to operate activity j
x_j	$=$	level of j-th activity, unknown, ≥ 0
b_i	$=$	when ≥ 0, resource amount of item i, otherwise delivery requirement
$b - Ax =: v$	≥ 0	amounts of items used in total do not exceed resources
y_i	$=$	prices of i-th item, unknown, ≥ 0
c_j	$=$	value of output of activity j on market
$A^T y - c =: u$	≥ 0	no activity makes positive profit

These relations are depicted in figure 3. That the costs for production of product j could only exceed the value of the product on the market is due to the (by assumption) perfect competition. The following conditions are quite intuitive:

$x \perp u$	no activity making negative profit is operated at level > 0
$z \perp v$	an item in excess supply has a zero price

The setting, all together, corresponds precisely to the LCP 0.1.1. $\max c^T x$ means to maximize the value of output of production and $\min y^T b$ is to minimize the cost for resources.

Lemma 1 (Lagrange Multiplier). *Consider the problem to find* $x \in \mathbb{R}^n$ *that* $\min f(x)$ *subject to* $g(x) \leq 0$ *and* $h(x) = 0$*, where* $g = (g_1, ..., g_m)^T$ *and* $h = (h_1, ..., h_k)^T$ *are in* $C^1(\mathbb{R}^n, \mathbb{R})$*, and* $k < n$*. Let* $\hat{x}$ *be a local minimum and regular, then there exist vectors* $\lambda \in \mathbb{R}^m$ *and* $\mu \in \mathbb{R}^k$ *so that*

$$\nabla_x L(\hat{x}, \lambda, \mu) := \nabla f + \lambda^T \nabla g + \mu^T \nabla h|_{\hat{x}} = 0,$$

$$\lambda \perp g(\hat{x}) \qquad \textit{and} \qquad \lambda \geq 0.$$

These relations are called the Kuhn-Tucker-conditions. The principle is illustrated in figure 4. When $m = 0$, the requirement of the gradient $\nabla_x L(\hat{x}, \lambda, \mu) = 0$ reduces to

$$\nabla f \in \operatorname{span}\{\nabla h_i \mid i = 1...k\}.$$

Remark 2 (Quadratic programming). Let all vectors and matrices be of appropriate dimensions and Q symmetric (why?). Consider the following problem:
Find $x \geq 0$ with $\min f(x) = \frac{1}{2} x^T Q x - c^T x$ where $b - Ax =: v \geq 0$. The constraints are concatenated in

$$g(x) = \begin{pmatrix} Ax - b \\ -x \end{pmatrix} \leq 0.$$

There are no strict equalities to satisfy, i.e. $k = 0$. Substituting $\lambda \to \binom{\lambda}{u}$, one easily derives from Lemma 1

$$\nabla_x L(x, \tbinom{\lambda}{u}, \mu)^T = Qx - c + (A^T \mid -I)\tbinom{\lambda}{u} = Qx - c + A^T\lambda - u = 0,$$

$$\begin{pmatrix} \lambda \\ u \end{pmatrix} \perp \begin{pmatrix} Ax - b \\ -x \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \lambda \\ u \end{pmatrix} \geq 0,$$

all together define the associated LCP

$$0 \leq \begin{pmatrix} x \\ \lambda \end{pmatrix} \perp \begin{pmatrix} u \\ v \end{pmatrix} = \left(\begin{array}{c|c} Q & A^T \\ \hline -A & 0 \end{array}\right) \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} -c \\ b \end{pmatrix} \geq 0.$$

Algorithm 1 (Solving the LCP in $\mathbf{O}(2^n)$). In order to find (all) $x \in \mathbb{R}^n$ satisfying

$$0 \leq x \perp Ax - b \geq 0,$$

we loop over all $I \in \mathfrak{P}(\{1, ..., n\})$. For such an index set I, let $J = \{1, ..., n\} \setminus I$, and we allow that $x_I \geq 0$ and set $x_J = 0$. Then

$$A_{I,I} x_I = b_I \tag{0.1.2}$$

is required by orthogonality. Furthermore,

$$A_{J,I} x_I \geq b_J. \tag{0.1.3}$$

Hence, if the matrix $A_{I,I}$ is invertible, we solve for x_I in 0.1.2 and simply check $x_I \geq 0$ and 0.1.3.
In case the matrix $A_{I,I}$ is singular we can still check whether there is an $\tilde{x}$ with $A_{I,I}\tilde{x} = b_I$. If so, the basis of the null space of $A_{I,I}$, denoted by $N = [\nu_1 \mid ... \mid \nu_{k>0}]$, with $A_{I,I}N = 0$ is of interest. The problem reduces to find a vector $\alpha \in \mathbb{R}^k$, which satisfies the remaining conditions $\tilde{x} + N\alpha =: x_I \geq 0$ and $A_{J,I}(\tilde{x} + N\alpha) \geq b_J$, joined into the linear program

$$\begin{pmatrix} N \\ A_{J,I}N \end{pmatrix} \alpha \geq \begin{pmatrix} -\tilde{x} \\ b_J - A_{J,I}\tilde{x} \end{pmatrix}.$$

Hence, if all submatrices $A_{I,I}$ are invertible, the solution space is a discrete subset of $\mathbb{R}^n$, possibly empty. On a today computer, the algorithm requires on average a day for a problem of dimension $n = 28$. If only one solution is of interest, reasonable heuristics might help to traverse $\mathfrak{P}(\{1, ..., n\})$.

Example 4 (Application of the rigorous algorithm). Consider the linear complementary problem instance where

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

$A_{I,I}$ is invertible for $I \in \{\emptyset, \{2\}, \{2,4\}, \{4\}\}$, and when $I = \{4\}$ we yield $x = (0\ 0\ 0\ 2)^T$. For $I = \{1, 2, 3\}$, the solution is of the form $x = (1\ 1\ x_3\ 0)^T$ with $x_3 \geq 0$. There are no other x solving the LCP, the solution space is non-discrete and disconnected.

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