■ Killing Fields in Negative Ricci Curvature

On a manifold M^m of dimension m we use the variables
• $p \in M$ for a point, • $f \in \mathfrak{F}$ as functions, $f : M \to \mathbb{R}$,
• $X, Y, Z, V, W \in \mathfrak{X}$ as vector fields, $X_p \in T_p M$,
• $\theta \in \mathfrak{X}^*$ as a one-form.
In the sequel, functions, vector fields, and forms are always smooth. X is **complete** if each of the maximal integral curves of X is defined over $\mathbb{R}$.

■ Summary of the semi-Riemannian manifold axiomatic

Let $(M^m, \langle\rangle)$ be a semi-Riemannian manifold. The (possibly local) **frame fields** are $E_i \in \mathfrak{X}$ for $i = 1, ..., m$ with $\langle E_i, E_j \rangle = \varepsilon^i \delta_{i,j}$.

The distinguished tensor field derivation **Lie derivative** L_X satisfies
l1 • $L_X f = X f$ l2 • $L_X Y = [X, Y]$ l3 • L_X is $\boldsymbol{R}$-linear in X

An affine connection $\nabla : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ written by $\nabla_Y X$ is
d1 • $\mathfrak{F}$-linear in Y d2 • $\boldsymbol{R}$-linear in X d3 • $\nabla_Y (f X) = (Y f) X + f \nabla_Y X$
The **Levi-Civita connection** $D = \nabla$ additionally satisfies uniquely
d4 • $[X, Y] = D_X Y - D_Y X$ d5 • $X \langle Y, Z \rangle - \langle D_X Y, Z \rangle - \langle Y, D_X Z \rangle = 0$
and is characterized by (Koszul)
d6 • $2 \langle D_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle$
$- \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle = 0$
dX is **parallel** if $D_Y X = 0$ for all Y.
D_X extends to a unique tensor field derivation (see ON.p64) with
d7 • $D_X f = X f$, and lhs of d5 is simply definition of $D_X \langle Y, Z \rangle$.

X is **Killing** supposing that
k1 • $L_X \langle Y, Z \rangle = X \langle Y, Z \rangle - \langle [X, Y], Z \rangle - \langle Y, [X, Z] \rangle = 0$
with d4 and d5

$$X \langle Y, Z \rangle - \langle D_X Y - D_Y X, Z \rangle - \langle Y, D_X Z - D_Z X \rangle = 0$$
$$\Longleftrightarrow \quad X \langle Y, Z \rangle - \langle D_X Y, Z \rangle + \langle D_Y X, Z \rangle - \langle Y, D_X Z \rangle + \langle Y, D_Z X \rangle = 0$$
$$\Longleftrightarrow \quad X \langle Y, Z \rangle - \langle D_X Y, Z \rangle - \langle Y, D_X Z \rangle + \langle D_Y X, Z \rangle + \langle Y, D_Z X \rangle = 0$$

k2 • $\langle D_Y X, Z \rangle + \langle Y, D_Z X \rangle = 0$
k3 • X parallel $\Longrightarrow$ X Killing, because k2 is satisfied:

$$\langle D_Y X, Z \rangle + \langle Y, D_Z X \rangle = \langle 0, Z \rangle + \langle Y, 0 \rangle = 0$$

Property d1 allows us to fix an X and define the tensor $D X \in \mathfrak{T}^1_1$ with

$$D X : \mathfrak{X} \to \mathfrak{X} \qquad Y \mapsto D_Y X, \text{ equivalently}$$
$$D X : \mathfrak{X}^* \times \mathfrak{X} \to \mathfrak{F} \qquad (\theta, Y) \mapsto \theta D_Y X$$

The iteration $(D X)^n$ is again in $\mathfrak{T}^1_1$, in particular

$$(D X)^2 : \mathfrak{X} \to \mathfrak{X} \qquad Y \mapsto D_{D_Y X} X$$

is $\mathfrak{F}$-linear in every slot as can be seen from

$$(D X)^2 : \mathfrak{X}^* \times \mathfrak{X} \to \mathfrak{F} \qquad (\theta, Y) \mapsto \theta D_{D_Y X} X$$

For X Killing, $D X$ is screw-symmetric by k2. As a consequence of definition PP.p166 the norm of $D X$ is the negative of the metrical contraction of $(D X)^2$ for

e1 • $| D X |^2 = \sum_i \varepsilon^i \langle D_{E_i} X, D_{E_i} X \rangle \overset{k2}{=} -\sum_i \varepsilon^i \langle D_{D_{E_i} X} X, E_i \rangle = -\mathsf{C}^1_1 (D X)^2$
e2 • $| D X |^2 = 0$ and all $\varepsilon^i = +1 \Longrightarrow \forall\, Y \;\; D_Y X = 0$

Some differential operators
gradient of f is grad $f \in \mathfrak{X}$ o1 • $\langle \text{grad} f, Y \rangle = Y f$
divergence of X is div $X \in \mathfrak{F}$ o2 • div $X = \sum \varepsilon^i \langle D_{E_i} X, E_i \rangle$
Hessian of f is $H^f \in \mathfrak{T}^0_2$ o3 • $H^f(Y, Z) = \langle D_Y \text{grad} f, Z \rangle$
Laplacian of $f \in \mathfrak{F}$ is $\Delta f \in \mathfrak{F}$ o4 • $-\Delta f = \text{div grad} f$, obviously
o5 • $-\Delta f \overset{o2}{=} \sum \varepsilon^i \langle D_{E_i} \text{grad} f, E_i \rangle = \sum \varepsilon^i H^f(E_i, E_i) = \mathsf{C}_{12} H^f$

Riemannian curvature tensor $R \in \mathfrak{T}^1_3$
c1 • $R_{X,Y} Z = D_{[X,Y]} Z - [D_X, D_Y] Z = D_{[X,Y]} Z - D_X D_Y Z + D_Y D_X Z$
c2 • $R_{X,Y} Z + R_{Y,Z} X + R_{Z,X} Y = 0$ (1st Bianchi)
Sectional curvature
c3 • $K(x, y) = \langle R_{x,y} x, y \rangle / (\langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle^2)$, which can be defined over M for two X, Y, with $X_p \notin \text{span}\, Y_p$ for all p. If $K(x, y) =$ const. one writes $K =$ const.
Ricci tensor Ric $\in \mathfrak{T}^0_2$ via contraction of R:
c4 • $\text{Ric}(X, Z) = \mathsf{C}^1_3 R_{X,Y} Z = \sum_i \varepsilon^i \langle R_{X,E_i} Z, E_i \rangle$
Scalar curvature via contraction of Ric.

s1 • $\int_{\partial M} \omega = \int_M d\omega$ (Stokes)
conclusion for manifolds without boundary, $\partial M = \emptyset$,
s2 • $0 = \int_{\partial M} \langle \nu, X \rangle \, d\partial M_g = \int_M \text{div}\, X \, dM_g$

■ Killing Fields in Negative Ricci Curvature

We investigate the amount of isometries of compact, orientable, connected Riemannian-manifolds with negative Ricci Curvature. Because of the relationship

$$\text{LA}(\text{Iso}\, M) = \mathfrak{iso}\, M = \text{Kill}_c\, M = \{X : X \text{ Killing and complete}\}$$

we might as well examine Killing vector fields on such manifolds to gain upper bounds for dimensions:

■ Utilities for [Bochner, 1946]

u1 • $\boxed{\text{grad} \tfrac{1}{2} \langle X, X \rangle = -D_X X}$; we test the vector field grad $\frac{1}{2} \langle X, X \rangle$ metrically with any Y
$\langle \text{grad} \tfrac{1}{2} \langle X, X \rangle, Y \rangle \overset{o1}{=} \tfrac{1}{2} Y \langle X, X \rangle \overset{d5}{=} \tfrac{1}{2} (\langle D_Y X, X \rangle + \langle X, D_Y X \rangle) = \langle D_Y X, X \rangle \overset{k2}{=} \langle -D_X X, Y \rangle$ □

For the next result we start manipulating
$H^{\frac{1}{2}\langle X,X \rangle}(Y, Z) \overset{o2}{=} \langle D_Y \text{grad} \tfrac{1}{2} \langle X, X \rangle, Z \rangle \overset{u1}{=} \langle D_Y(-D_X X), Z \rangle \overset{d2}{=} \langle -D_Y D_X X, Z \rangle$
with c1 $R_{X,Y} X = D_{[X,Y]} X - D_X D_Y X + D_Y D_X X$, hence the vector field

$$\begin{aligned} -D_Y D_X X &\overset{c1}{=} -R_{X,Y} X + D_{[X,Y]} X - D_X D_Y X \\ &\overset{d4}{=} -R_{X,Y} X + D_{D_X Y - D_Y X} X - D_X D_Y X \\ &\overset{d1}{=} -R_{X,Y} X + D_{D_X Y} X - D_{D_Y X} X - D_X D_Y X \end{aligned}$$

Together
u2 • $\boxed{H^{\frac{1}{2}\langle X,X \rangle}(Y, Z) = \langle -R_{X,Y} X - D_{D_Y X} X + D_{D_X Y} X - D_X D_Y X, Z \rangle}$ □

Contracting every term of u2 gives
u3 • $\boxed{-\Delta \tfrac{1}{2} \langle X, X \rangle = -\text{Ric}(X, X) + | D X |^2}$
First the lhs

$\sum \varepsilon^i H^{\frac{1}{2}\langle X,X\rangle}(E_i, E_i) \overset{\text{o5}}{=} -\Delta \frac{1}{2} \langle X, X\rangle$,
the next term by definition of Ricci curvature
$-\sum_i \varepsilon^i \langle R_{X,E_i} X, E_i\rangle \overset{\text{c3}}{=} -\text{Ric}(X, X)$,
then the definition of the operator norm
$-\sum_i \varepsilon^i \langle D_{D_{E_i} X} X, E_i\rangle \overset{\text{e1}}{=} |D X|^2$. It remains to show that
$\sum_i \varepsilon^i \langle D_{D_X E_i} X - D_X D_{E_i} X, E_i\rangle \overset{!}{=} 0$, or an even stronger statement
$\langle D_{D_X Y} X - D_X D_Y X, Y\rangle = \langle D_{D_X Y} X, Y\rangle - \langle D_X D_Y X, Y\rangle$
$\quad \overset{\text{k2}}{=} -\langle D_Y X, D_X Y\rangle - \langle D_X D_Y X, Y\rangle \overset{\text{d5}}{=} D_X \langle D_Y X, Y\rangle \overset{\text{k2}}{=} D_X 0 = 0$
for all Y. □

■ **[Bochner, 1946]** $(M^m, \langle\rangle)$ compact, oriented, Riemannian metric, and X Killing
b1 • $\boxed{\text{Ric}(Y, Y) \le 0 \text{ for all } Y \Longrightarrow X \text{ parallel}}$
Via integration over tensor norm we see
$0 \overset{\text{s2}}{=} \int_M \text{div grad} \frac{1}{2} \langle X, X\rangle \overset{\text{def}}{=} \int_M -\Delta \frac{1}{2} \langle X, X\rangle \, dM_g$
$\quad \overset{\text{u3}}{=} \int_M -\text{Ric}(X, X) + |D X|^2 \, dM_g \ge \int_M |D X|^2 \, dM_g \ge 0$
$\Longrightarrow |D X|^2 = 0$, i.e. $D_Y X = 0$ for all $Y \overset{\text{def}}{\Longrightarrow} X$ is parallel. □
b2 • $\text{Ric} < 0 \overset{\text{def}}{\Longleftrightarrow} \boxed{\text{Ric}(Y, Y) \begin{cases} < 0 & \text{where } Y \ne 0 \\ = 0 & \text{where } Y = 0 \end{cases} \Longrightarrow X = 0}$
since we have $\text{Ric}(X, X) = 0 \Longrightarrow X = 0$. □

■ **[Gao & Yau, 1986]** On every compact $M^3 \Longrightarrow \exists$ Riemannian metric so that $\text{Ric} \le 0$.

■ **Utilities for [Corollary 1]**

κ1 • $\mathfrak{iso}\, M = \text{Kill}_c\, M = \{X : X \text{ Killing and complete}\}$
κ2 • M connected $\Longrightarrow$ the map $\varrho_p : \text{Kill}_c\, M \to T_p M \times \mathfrak{so}(T_p M)$ with $X \mapsto (X_p, (D X)_p)$ is injective, meaning a Killing and complete X is determined by X_p and $(D X)_p$ for an arbitrary p. Since $\dim T_p M = m$ and $\dim \mathfrak{so}(\mathbb{R}^m) = \dim \{A \in \mathbb{R}^{m\times m} : A^T = -A\} = \frac{m(m-1)}{2}$ we estimate
$\quad \dim \text{Iso}\, M = \dim \mathfrak{iso}\, M = \dim \text{Kill}_c\, M \le m + \frac{m(m-1)}{2}$
γ1 • M compact $\Longrightarrow \text{Iso}\, M$ compact

■ **[Corollary 1]** $(M^m, \langle\rangle)$ compact, oriented, Riemannian metric, connected, then
b3 • $\boxed{\ldots \text{Ric} < 0 \Longrightarrow \dim \text{Iso}\, M = \dim \mathfrak{iso}\, M \le \dim M}$
We consider the Killing Fields since
$\quad \dim \mathfrak{iso}\, M \overset{\kappa 1}{=} \dim \text{Kill}_c\, M = \{X : X \text{ Killing and complete}\} \qquad (*)$
and by κ2 (using connectedness) the map
$\quad \varrho_p : \text{Kill}_c\, M \to T_p M \times \mathfrak{so}(T_p M)$ with $X \mapsto (X_p, (D X)_p)$
is injective. Each Killing X is parallel, meaning $D_Y X = 0$. The value of a tensor field at p depends only on the input values at p. We write $0 = (D_Y X)_p = (D_{Y_p} X) = (D_y X)_p$ for all $y \in T_p$, hence the restriction
$\quad \overline{\varrho}_p : \text{Kill}_c\, M \to T_p M$ with $X \mapsto X_p$
is already injective. We continue with (*)
$\quad \dim \text{Kill}_c\, M \le \dim T_p M = \dim M$. □
b4 • $\boxed{\ldots \text{Ric} < 0 \Longrightarrow |\text{Iso}\, M| < \infty}$
For Ric < 0 by b2 Killing fields vanish, i.e.
$\quad \dim \text{Kill}_c\, M = \dim \{0\, \mathfrak{X}\} = \dim \mathfrak{iso}\, M = \dim \text{Iso}\, M = 0$.
Due to compactness $\text{Iso}\, M$ is not only countable but also finite. □

■ **[Corollary 2]** $(M^m, \langle\rangle)$ compact, oriented, Riemannian metric, connected, and **quasi-negative Ricci curvature**, i.e.

Ric ≤ 0 and $\mathrm{Ric}(y, y) < 0$ for all $y \in T_p M \setminus 0$ for some p, then

b5 • $\boxed{\ldots X \text{ Killing} \Longrightarrow X = 0}$

assume: X Killing $\wedge$ $X \neq 0$ $\Longrightarrow$ X parallel $\wedge$ $X \neq 0 \Longrightarrow X$ vanishes nowhere $\Longrightarrow$ for some p $\mathrm{Ric}(X_p, X_p) < 0$, but we already argued above that $\mathrm{Ric}(X, X) = 0$. □

■ **References**

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