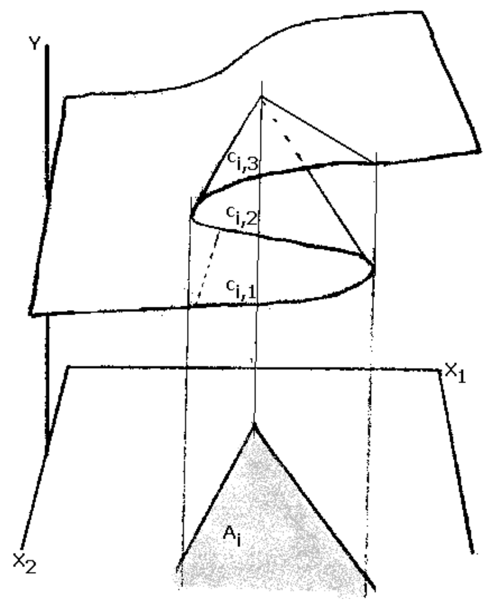

Figure 1 : $\xi_{i,j}$ are denoted $c_{i,j}$; lines might not be as straight as they appear.

0.1 Preliminary

Thoughout the text R is a real closed field. When we write "homeomorphic", we mean "semi-algebraically homeomorphic".

Proposition 1 (1.4.6). *Let $X = (X_1, ..., X_n)$ and $f_1(X, Y), ..., f_s(X, Y)$ be polynomials in $n + 1$ variables, ranging over R with coefficients in $\mathbb{Z}$ and $\omega \in W_{s,q}$, where q is the maximum degree in Y of the s polynomials. There exists a boolean combination $\mathcal{B}_\omega(X)$ of polynomial equations and inequalities in the variables $X = (X_1, ..., X_n)$ with coefficients in $\mathbb{Z}$, so that for every $x \in R^n$, we have*

$$\mathcal{B}_\omega(x) = \top \qquad \Longleftrightarrow \qquad SIGN_R[f_1(x, Y), ..., f_s(x, Y)] = \omega.$$

Proposition 2 (2.1.7). *Semi-algebraic subsets of R are exactly the finite unions of points and open intervals (bounded or unbounded).*

Proposition 3 (2.2.4). *Let $\Phi(x_1, ..., x_n)$ be a first-order formula of the language of ordered fields, with parameters in R and free variables $x_1, ..., x_n$. Then $\{x \in R^n \mid \Phi(x) = \top\}$ is a semi-algebraic set.*

Definition 1 (2.2.5). Let $A \subset R^m$ and $B \subset R^n$ be two semi-algebraic sets. A mapping $f : A \to B$ is *semi-algebraic* if its graph is semi-algebraic in R^{m+n}.

0.2 Decomposition of semi-algebraic sets I

A semi-algebraic set $S \subset R^n$ is the intersection and union of sets of the form $\{x \in R^n \mid f(x) * 0\}$, where $f(X)$ is a polynomial in n variables $X = (X_1, ..., X_n)$ with coefficients in R, and $*$ stands for $=$ or $<$. Our goal is to show, that S is the disjoint union of finitely many subsets of R^n each homeomorphic to a hypercube $]0,1[^d \subset R^d$ for some $d \in \mathbb{N}_0$ ($]0,1[^0$ stands for $\{0\}$). The approach is almost of constructive nature.

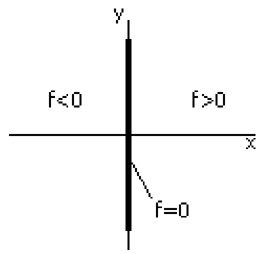

Figure 2 : Exciting implicit plot of $f(X,Y) = X_1$.

Example 1. Let $X = (X_1, X_2) \in R^2$ and consider $f(X,Y) = Y^3 - YX_2 - X_1$. The surface defined by $f = 0$ is depicted in figure 1. Using f we may define semi-algebraic sets such as $M_\# := \{(x_1, x_2, y) \in R^3 \mid f(x,y) * 0\}$, where $*$ stands for an operator out of $=, <, \leq, ...$.

Definition 2 (Slicing). Let $f_1(X,Y), ..., f_s(X,Y)$ be polynomials in the $n+1$ variables $X_1, ..., X_n, Y$ obtaining values in R with coefficients in R. A partition of R^n into a finite number of semi-algebraic sets $A_1, ..., A_m$ and, for $i = 1...m$, a finite number l_i (possibly zero) of continuous semi-algebraic functions $\xi_{i,1} < ... < \xi_{i,l_i}$, $\xi_{i,j} : A_i \to R$, so that

- for every $x \in A_i$, $\{\xi_{i,1}(x), ..., \xi_{i,l_i}(x)\}$ is the set of roots of those polynomials among $f_1(x,Y), ..., f_s(x,Y)$, which are not identically zero, and
- for all $x \in A_i$, $\text{SIGN}_R[f_1(x,Y), ..., f_s(x,Y)]$ is invariant,

is called a *slicing* of $f_1, ..., f_s$ and denoted by $(A_i, (\xi_{i,j})_{j=1...l_i})_{i=1...m}$.

Later, in theorem 5, we will prove that such a slicing always exists.

Remark 1. Consider $f(X,Y) = X_1$ with $n = s = 1$ as illustrated in figure 4. The second condition ensures that R is partitioned into $]-\infty, 0[, \{0\},]0, \infty[$, instead of simply R.

Theorem 4 (Decomposition I). *Every semi-algebraic subset of R^n is the disjoint union of a finite number of semi-algebraic sets, each of which is homeomorphic to an open hypercube $]0,1[^d \subset R^d$ for some $d \in \mathbb{N}_0$.*

Proof. By induction on n. The base case $n = 1$ is clear, since by proposition 2 every semi-algebraic subset of R is the union of a finite number of points and open intervals. Sets which have a non-empty intersection, can be joined to points or open intervals, until we are left with pairwise disjoint sets, each homeomorphic to $\{0\}$ or $]0,1[$.
Assume claim proved up to n. Let $S \subset R^{n+1}$ be semi-algebraic, given by a boolean combination of sign conditions on the polynomials $f_1(X,Y), ..., f_s(X,Y)$ with coefficients in R sliced into $(A_i, (\xi_{i,j})_{j=1...l_i})_{i=1...m}$. We can write S as the union of sets that are either

- the graph of $\xi_{i,j} \simeq A_i$, or
- a slice $]\xi_{i,j}, \xi_{i,j+1}[:= \{(x,y) \in A_i \times R \mid \xi_{i,j}(x) < y < \xi_{i,j+1}(x)\} \simeq A_i \times \,]0,1[$, for $j = 0...l_i$ while setting $\xi_{i,0} \equiv -\infty$ and $\xi_{i,l_i+1} \equiv \infty$ over A_i.

Either case defines a semi-algebraic set in R^{n+1}. There is only a finite number of those. Since all A_i are semi-algebraic, then by assumption, each A_i is the disjoint union of a finite number of semi-algebraic sets, each homeomorphic to an open hypercube $]0,1[\subset R^d$ for some $d \in \mathbb{N}_0$. Consequently, we easily construct the required sets of the claim. □

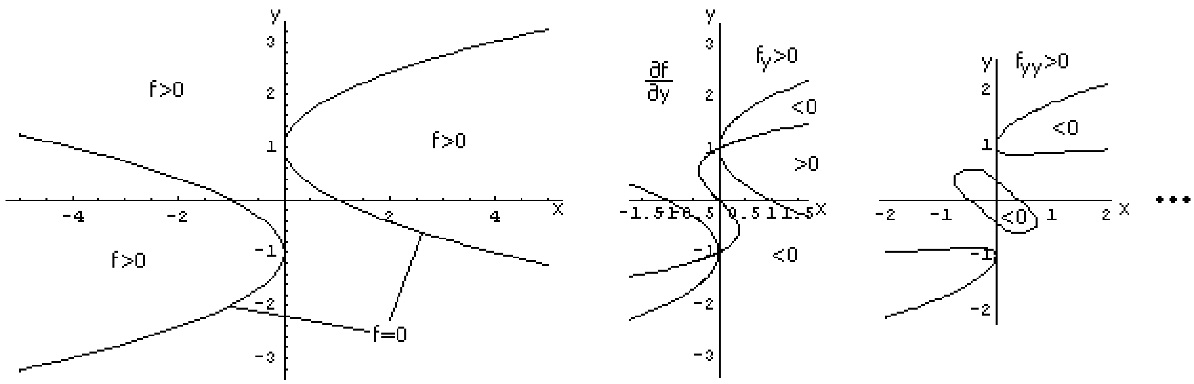

Figure 3 : Adding the 8 non-vanishing derivatives of f from example 3 with respect to Y to the list of polynomials results in a finer partition, which finally slices f.

Remark 2. The claim "$]\xi_{i,j}, \xi_{i,j+1}[\ \simeq A_i \times\ ]0,1[$" follows by constructing explicitely a homeomorphism

$$h\colon\ ]\xi_{i,j}, \xi_{i,j+1}[\ \to A_i \times\ ]0,1[,$$

requiring $\xi_{i,j} \neq \xi_{i,j+1}$ everywhere, and both functions $\xi_\# : A_i \to R$ being continuous. The proof uses for example a continuous bijective function $g\colon R =]-\infty, \infty[\ \to\ ]0,1[$ given by

$$g(y) := \frac{y + \sqrt{1+y^2}}{2\sqrt{1+y^2}}.$$

Theorem 5 (Existence of slicing). *A slicing as defined in definition 2, always exists.*

Proof. For the moment, assume that the coefficients of the polynomials $f_1(X,Y), ..., f_s(X,Y)$ are from $\mathbb{Z}$, with q being the maximum degree in Y of the f_k's. Let $\omega \in W_{s,q}$. According to Proposition 1.4.6, there exists a boolean combination $\mathcal{B}_\omega(X)$ of polynomial equations and inequalities in the variables $X = (X_1, ..., X_n)$ with coefficients in $\mathbb{Z}$, so that for every $x \in R^n$, we have

$$\mathcal{B}_\omega(x) = \top \qquad \Longleftrightarrow \qquad \mathrm{SIGN}_R[f_1(x,Y), ..., f_s(x,Y)] = \omega.$$

To partition R^n into A_i we loop over all $\omega \in W_{s,q}$, and if $A_\omega := \{x \in R^n \mid \mathcal{B}_\omega(x)\} \neq \varnothing$ we make A_ω a set of the partition. There are only finitely many of those, say $A_{i=1...m}$. Then each A_i is semi-algebraic and disjoint to all other $A_{j\neq i}$. Together the $A_{i=1...m}$ cover R^n, thus they form a finite partition of R^n.

Example 2. Again, consider $f(X_1, Y) = X_1$ with $n = s = 1$. The maximum degree in Y is $q = 0$. Letting ω range over $W_{1,0} = \{(-1),(0),(1)\}$, we obtain the partition $]-\infty, 0[, \{0\},]0, \infty[$.

Example 3 (Pathology). Let $f(X_1, Y) = (X_1 - (Y-1)^2)^2 (X_1 + (Y+1)^2)^2$, we yield that $\mathrm{SIGN}_R[f(x,Y)] = (1\ 0\ 1\ 0\ 1)$ for all $x \in R$, thus R is partitioned into R. But it is impossible to find two continuous semi-algebraic functions $\xi_1 < \xi_2$ defined on R giving the roots of $f(x,Y)$. The subdivision of the domain into $]-\infty, 0[, \{0\},]0, \infty[$ would work, which is clear from figure 3.

We continue with the proof of theorem 5. The previous example motivates the necessity of adding the non-vanishing derivatives of each $f_1, ..., f_s$ with respect to Y to the list of polynomials. In the following we will assume that $f_1, ..., f_s$ is stable under derivation with respect to Y.

$\text{SIGN}_R[f_1(x,Y),...,f_s(x,Y)]$ is constant for all $x \in A_i$, then there is a number $l_i \leq sq$ so that the polynomials among $f_1(x,Y),...,f_s(x,Y)$ that are not identical to zero have together l_i roots $\xi_{i,1}(x) < ... < \xi_{i,l_i}(x)$. The graph of $\xi_{i,j} : A_i \to R$ is

$$\{(x,y) \in A_i \times R \mid \exists (y_1...y_{l_i}) \in R^{l_i} \ [\prod_k f_k(x,y_1) = ... = \prod_k f_k(x,y_{l_i}) = 0 \text{ and } y_1 < ... < y_{l_i} \text{ and } y_j = y]\}.$$

According to proposition 2.2.4 and definition 2.2.5 $\xi_{i,j}$ is a semi-algebraic mapping.
Let for a fixed $x' \in A_i$ be $f_{p_1}(x',Y),...,f_{p_{l_i}}(x',Y)$ polynomials that have simple zero in $\xi_{i,1}(x') < ... < \xi_{i,l_i}(x')$. These polynomials exist, because we have previously added the Y-derivatives. The inequality

$$f_{p_q}(x',\xi_{i,q}(x') - \epsilon) f_{p_q}(x',\xi_{i,q}(x') + \epsilon) < 0$$

holds for $\epsilon > 0$ small enough and all $q = 1...l_i$. The inequality remains satisfied when substituting for x' an x out of a small enough environment of x' in R^n. Such an environment can be established so that the inequality holds for all $q = 1...l_i$ simultaneously, which proves that the $\xi_{i,q}$ are continous.
Functions $\xi_{i,j}$ that do not give roots of polynomials of the initial family are removed.
Now, let the coefficients of $f_1(X,Y),...,f_s(X,Y)$ be in R. We perform the following transformation [**Lemma 2.3.2**]: Each coefficient becomes a new variable, we design polynomials $\bar{f}_1(A,X,Y),...,\bar{f}_s(A,X,Y)$ with coefficients in $\mathbb{Z}$, so that, if $\bar{a} = (\bar{a}_1,...,\bar{a}_c) \in R^c$ is the concatenation of the coefficients of all the polynomials $f_{k=1...s}$, the identity $\bar{f}_k(\bar{a},X,Y) = f_k(X,Y)$ holds for all $k = 1...s$. Proposition 1.4.6 applies to the new $\bar{f}_1(A,X,Y),...,\bar{f}_s(A,X,Y)$: Let $\omega \in W_{s,q}$, where q is the maximum degree in Y of the s polynomials. There exists a boolean combination $\bar{\mathcal{B}}_\omega(A,X)$ of polynomial equations and inequalities in the variables (A,X) with coefficients in $\mathbb{Z}$, so that for every $(a,x) \in R^{c+n}$, we have

$$\bar{\mathcal{B}}_\omega(a,x) = \top \qquad \Longleftrightarrow \qquad \text{SIGN}_R[f_1(a,x,Y),...,f_s(a,x,Y)] = \omega.$$

In our construction above simply substitute $\mathcal{B}_\omega(X) = \bar{\mathcal{B}}_\omega(\bar{a},X)$. This concludes the proof. □

0.3 Connectedness; decomposition of semi-algebraic sets II

Our goal is to show that a semi-algebraic set $S \subset R^n$ is the disjoint union of a finite number of semi-algebraically connected semi-algebraic sets $C_1,...,C_s$, which are both open and closed in S.
We establish the topological space $(R^n,\mathcal{O})$, where the collection of open sets $\mathcal{O}$, with $\{R^n,\varnothing\} \subset \mathcal{O}$, is generated by finite intersections and arbitrary unions of sets of the form

$$B_r(x) = \{y \in R^n \mid ||y - x|| < r\}, \qquad x \in R^n,\ r \in R_{>0},$$

with the standard norm, or, for $n = 1$ equivalently generated by

$$]-\infty,r[\qquad \text{and} \qquad]r,\infty[\qquad \forall r \in R.$$

The closure of set $S \subset R^n$ is defined as $\bar{S} := \bigcap\{A \subset R^n \mid S \subset A \text{ and } R^n \setminus A \in \mathcal{O}\}$.

Example 4. In $(R = \mathbb{R}_{alg},\mathcal{O})$ the set $P_- = \bigcup_{r\in R_{<\pi}}]-\infty,r[$ is open. The complement of P_- is $P_+ = R \setminus P_- = \bigcup_{r\in R_{>\pi}}]r,\infty[$ is open as well. Because $P_- \cap P_+ = \varnothing$ and $P_- \cup P_+ = R$ but $P_- \neq R$ and $P_- \neq \varnothing \implies$ R is not connected according to the standard definition of connectedness. P_- and P_+ are closed but not semi-algebraic in $\mathbb{R}_{alg}$.
In $\mathbb{R}(X)^\wedge$ the set $\{f \in \mathbb{R}(X)^\wedge \mid \exists r \in \mathbb{R}\ r > 0 \text{ and } f > r\}$ is a closed and open set.

Example 4 motivates the following definition for connectedness.

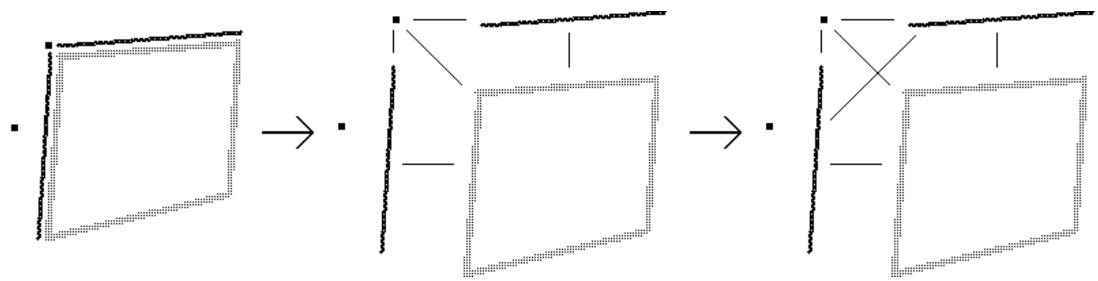

Figure 4 : Left: decomposition $B_{i=1...n}$, intermediate: $B_i \cap \bar{B}_j \neq \varnothing$, right: equivalence relation

Definition 3 (Connectedness). A semi-algebraic subset $S \subset R^n$ is *semi-algebraic connected* in $(R^n, \mathcal{O})$ if for every pair of semi-algebraic sets F_1 and F_2 open in $(S, \mathcal{O} \cap S)$, one has

$$F_1 \cap F_2 = \varnothing \qquad F_1 \cup F_2 = S \qquad \Longrightarrow \qquad F_1 = S \text{ or } F_1 = \varnothing.$$

In the sequel: "connected" means "semi-algebraically connected".

Remark 3. Let f be a semi-algebraic continous function. The image $f(C)$ of a connected set C is connected.

Proposition 6. *An open hypercube* $]0,1[^d \subset R^d$ *is connected.*

Proof. For $d = 1$, $]0,1[\subset R$ is connected. Take semi-algebraic sets $F_1, F_2 \subset]0,1[$ and open in $]0,1[$ with $F_1 \cap F_2 = \varnothing$ and $F_1 \cup F_2 =]0,1[$. F_1 semi-algebraic and open implies by proposition 2.1.7, that $F_1 = \bigcup^n]a_i, b_i[$ for appropriate $a_i, b_i \in [0,1]$. F_2 has to be the complement of F_1 in $]0,1[$, i.e. $F_2 = \bigcap^n]0, a_i] \cup [b_i, 1[$ and F_2 open implies $F_2 =]0,1[$ or empty.
Now let $d > 1$: Assume "not", then there exist open, non-empty $F_1, F_2 \subset R^d$ that partition $]0,1[^d \subset R^d$. Choose $x_1 \in F_1$ and $x_2 \in F_2$. Denote with h the homeomorphism $h(\lambda) = \lambda x_1 + (1-\lambda)x_2$, that maps $]0,1[$ bijectively to the segment $\Lambda :=]x_1, x_2[$. With $F_1' = \Lambda \cap F_1$ and $F_2' = \Lambda \cap F_2$ the set $\Lambda = h(]0,1[)$ is disconnected - a contradiction to remark 3. □

Theorem 7 (Decomposition II). *Every semi-algebraic subset $S \subset R^n$ is the disjoint union of a finite number of connected semi-algebraic sets $C_1, ..., C_s$, which are both closed and open in S. The $C_1, ..., C_s$ are called the semi-algebraically connected components of S.*

Proof. Denote with $B_1, ..., B_n$ the finite partition of S into semi-algebraic sets, B_i homeomorphic to $]0,1[^d$ for some d. Consider the equivalence relation generated by

$$B_i \sim B_j \qquad \Longleftrightarrow \qquad B_i \cap \bar{B}_j \neq \varnothing.$$

Let there be s equivalence classes and C_k be the union of all B_i in the k-th class. The C_k are semi-algebraic and open in S. Also, they form another partition of S. Suppose $C_k = F_1 \cup F_2$ for disjoint, semi-algebraic F_1, F_2 open in C_k. Since each B_i is connected,

$$B_i \subset C_k \qquad \Longrightarrow \qquad B_i \subset F_1 \text{ or } B_i \subset F_2.$$

If $B_i \subset F_1$ (resp. F_2) and $B_i \cap \bar{B}_j \neq \varnothing \Longrightarrow B_j \subset F_1$ (resp. F_2). According to the definition of the C_k, we have $C_k = F_1$ or $C_k = F_2$. □

References:

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