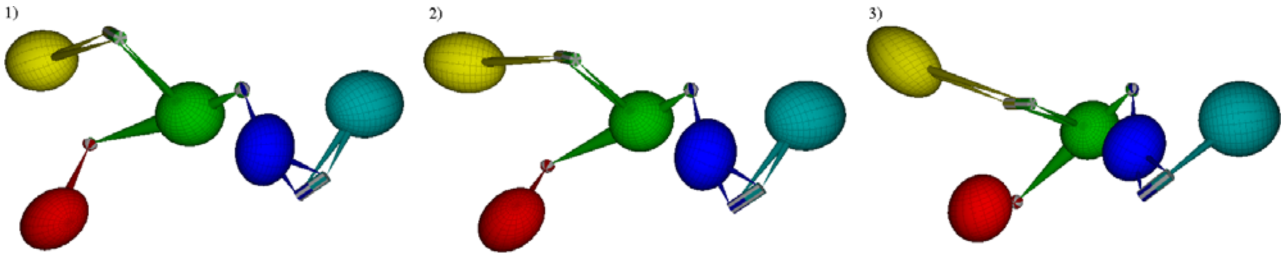
Animation of Skeletons with Hinges and Spherical Joints

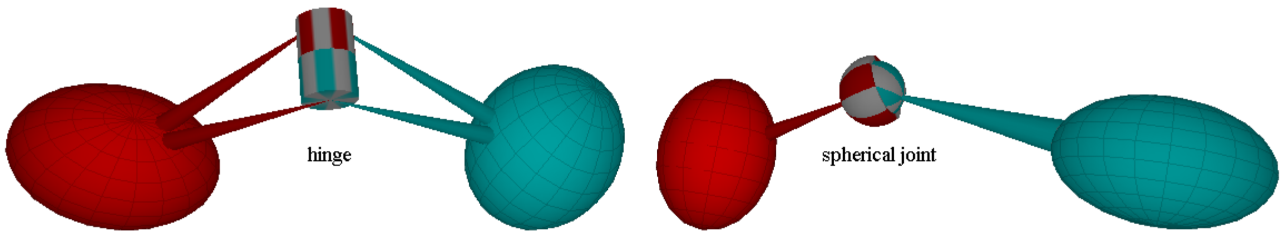
A derivation by Jan Hakenberg dedicated to Nikolai Sperling.



Abstract: We derive an algorithm to animate a skeleton of rigid bodies that are linked by hinges and spherical joints. Over the course of the simulation, the total linear momentum, and the total angular momentum are invariant. If desired, the algorithm incorporates intrinsic torques of the joints such as friction, and motor control. Otherwise, the total kinetic energy is invariant, too.

■ Types of joints

We intend to animate skeletons of rigid bodies that are linked by any combination of hinges and spherical joints. A hinge has a distinct axis of rotation, around which the attached bodies rotate relative to each other. Therefore, we visualize a hinge by a cylinder. The levers of a spherical joint are free to revolve around any axis relative to each other. We represent a spherical joint by a sphere.

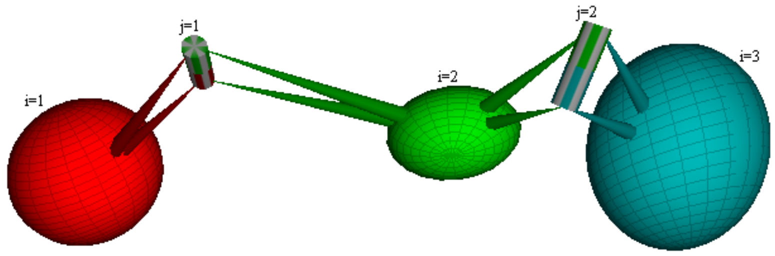


Our article starts by considering skeletons with hinges only. However, the introduction of spherical joints turns out to be simple at a later point: the vector that formerly represented the axis of a hinge is just set to zero.

■ Topology of the skeleton

The skeleton shall consist of $n+1$ rigid bodies that are pairwise linked by n hinges. (Later, any hinge can be replaced by a spherical joint.) The topology of the skeleton resembles a tree, i.e. a connected graph with no circles. We enumerate the bodies with the index $i = 1, 2, \ldots, n+1$, and we enumerate the hinges using $j = 1, 2, \ldots, n$. The connectivity of the skeleton is encoded in the $(n, 2)$-matrix E, where row j of E contains the indices of the two bodies connected by hinge j. From E we construct the $(n, n+1)$-matrix σ with entries as

$$\sigma_{j,i} = \begin{cases} +1 & \text{for } i = E(j, 1) \\ -1 & \text{for } i = E(j, 2) \\ 0 & \text{otherwise} \end{cases}$$



Example: The skeleton depicted above serves as an example throughout the document. The skeleton consists of 3 bodies and $n = 2$ hinges. Using the indexing as in the graphics, the topology is encoded by

$$E = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \qquad \text{and} \qquad \sigma = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

■ Generic equations of motion

The state of the body $i = 1, 2, \ldots, n+1$ is defined by the following variables:

p_i	center of mass in world coordinates • (3)-vector
R_i	orientation transforming from object to world coordinates • orthogonal (3,3)-matrix
v_i	linear velocity in world coordinates • (3)-vector
ω_i	angular velocity in object coordinates • (3)-vector
I_i	inertia tensor • constant symmetric (3,3)-matrix
m_i	mass • constant real value, greater than zero

The dynamics of each body i are determined by

a_i	linear acceleration in world coordinates • (3)-vector
τ_i	torque in object coordinates • (3)-vector

The entities p_i, R_i, v_i, ω_i, a_i, τ_i depend on time t, while the inertia and mass I_i, m_i are assumed to be constant. The linear motion results from the differential equations

$$d_t\, v_i = a_i$$
$$d_t\, p_i = v_i$$

The angular motion is governed by the differential equations

$$d_t\, \omega_i = I_i^{-1}.(-\Omega_i.I_i.\omega_i + \tau_i)$$
$$d_t\, R_i = R_i.\Omega_i$$

where Ω is a skew-symmetric (3,3)-matrix composed of the three entries of ω as

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

Example: In the illustrations, the ellipsoids visualize the inertia tensor I_i. The extensions of the shape correspond to the three eigenvalues of I_i. The center of mass p_i is located in the center of the ellipsoid.

■ Constraints by hinges

The location and alignment of the hinge $j = 1, 2, \ldots, n$ is constant with respect to the two bodies $E(j, 1)$, and $E(j, 2)$, the hinge connects. We define

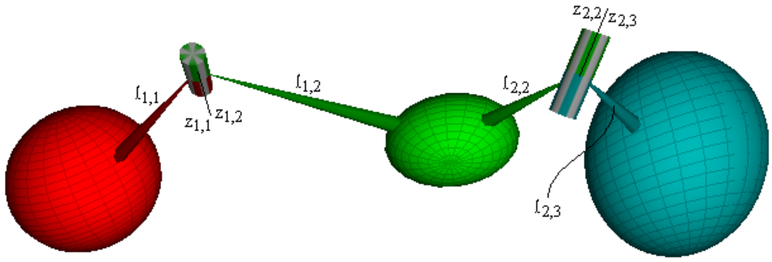
$\hat{l}_{j,1}$ location of hinge j in object coordinates of body $E(j, 1)$ • constant (3)-vector
$\hat{l}_{j,2}$ location of hinge j in object coordinates of body $E(j, 2)$ • constant (3)-vector

and

$\hat{z}_{j,1}$ axis of hinge j in object coordinates of body $E(j, 1)$ • constant (3)-vector of norm 1
$\hat{z}_{j,2}$ axis of hinge j in object coordinates of body $E(j, 2)$ • constant (3)-vector of norm 1

For the purpose of enumeration, we define additional vectors $l_{j,i}$ for $i = 1, 2, \ldots, n+1$ via

$$l_{j,i} = \begin{cases} \hat{l}_{j,1} & \text{for } i = E(j, 1) \\ \hat{l}_{j,2} & \text{for } i = E(j, 2) \\ 0 & \text{otherwise} \end{cases} \qquad z_{j,i} = \begin{cases} \hat{z}_{j,1} & \text{for } i = E(j, 1) \\ \hat{z}_{j,2} & \text{for } i = E(j, 2) \\ 0 & \text{otherwise} \end{cases}$$



Example: For joint $j = 2$ of the skeleton depicted above, we have $l_{2,1} = 0$, but $l_{2,2} = \hat{l}_{2,1}$, and $l_{2,3} = \hat{l}_{2,2}$. Analogous, $z_{2,1} = 0$, but $z_{2,2} = \hat{z}_{2,1}$, and $z_{2,3} = \hat{z}_{2,2}$. ■

Henceforth, we use the shorthand $\text{j1} = E(j, 1)$ and $\text{j2} = E(j, 2)$. For instance, $p_{\text{j1}} = p_{E(j,1)}$ and $z_{j,\text{j2}} = z_{j,E(j,2)}$. The hinges shall not separate over the course of the simulation. At any time t, we demand

(1) $$p_{\text{j1}} + R_{\text{j1}}.l_{j,\text{j1}} = p_{\text{j2}} + R_{\text{j2}}.l_{j,\text{j2}} \qquad \text{for } j = 1, 2, \ldots, n$$
$$R_{\text{j1}}.z_{j,\text{j1}} = R_{\text{j2}}.z_{j,\text{j2}}$$

The time derivatives d_t of these equations are

(2) $$v_{\text{j1}} + R_{\text{j1}}.\Omega_{\text{j1}}.l_{j,\text{j1}} = v_{\text{j2}} + R_{\text{j2}}.\Omega_{\text{j2}}.l_{j,\text{j2}} \qquad \text{for } j = 1, 2, \ldots, n$$
$$R_{\text{j1}}.\Omega_{\text{j1}}.z_{j,\text{j1}} = R_{\text{j2}}.\Omega_{\text{j2}}.z_{j,\text{j2}}$$

Finally, another application of d_t yields the relations

(3) $$a_{\text{j1}} + R_{\text{j1}}.\Omega_{\text{j1}}.\Omega_{\text{j1}}.l_{j,\text{j1}} - R_{\text{j1}}.L_{j,\text{j1}}.d_t\,\omega_{\text{j1}} = a_{\text{j2}} + R_{\text{j2}}.\Omega_{\text{j2}}.\Omega_{\text{j2}}.l_{j,\text{j2}} - R_{\text{j2}}.L_{j,\text{j2}}.d_t\,\omega_{\text{j2}} \qquad \text{for } j = 1, 2, \ldots, n$$
$$R_{\text{j1}}.\Omega_{\text{j1}}.\Omega_{\text{j1}}.z_{j,\text{j1}} - R_{\text{j1}}.Z_{j,\text{j1}}.d_t\,\omega_{\text{j1}} = R_{\text{j2}}.\Omega_{\text{j2}}.\Omega_{\text{j2}}.z_{j,\text{j2}} - R_{\text{j2}}.Z_{j,\text{j2}}.d_t\,\omega_{\text{j2}}$$

where we have substituted the cross product by vector $l_{j,i}$, $z_{j,i}$ with the skew-symmetric (3,3)-matrix $L_{j,i}$, $Z_{j,i}$ composed as

$$L_{j,i} = \begin{pmatrix} 0 & -l_{j,i_3} & l_{j,i_2} \\ l_{j,i_3} & 0 & -l_{j,i_1} \\ -l_{j,i_2} & l_{j,i_1} & 0 \end{pmatrix} \quad \text{and} \quad Z_{j,i} = \begin{pmatrix} 0 & -z_{j,i_3} & z_{j,i_2} \\ z_{j,i_3} & 0 & -z_{j,i_1} \\ -z_{j,i_2} & z_{j,i_1} & 0 \end{pmatrix}$$

The relations (1), and (2) are meaningful constraints on the initial configuration of the skeleton: the animation launches with the hinges touching and well aligned, and not about to be torn apart. From (3), we derive linear accelerations and torques on the $n+1$ bodies that ensure (1) and (2) over the course of the animation.

▪ Dynamic impact at hinges

To model friction and motor control at joint j, we introduce

e_j intrinsic torque at joint j in world coordinates • (3)-vector

If joint j revolves frictionless and passively, we set $e_j = 0$. In general, we propose the expression

$$e_j = -\mu_j\,[R_{\text{j1}}.\omega_{\text{j1}} - R_{\text{j2}}.\omega_{\text{j2}}] + \beta_j\,R_{\text{j1}}.z_{j,\text{j1}}$$

where $\mu_j \geq 0$ is the friction coefficient, and β_j is the torque of the motor attached to hinge j.

In each timestep, we compute (3)-vectors c_j, d_j in world coordinates for all hinges $j = 1, 2, \ldots, n$. The vectors c_j, d_j shall contribute to the linear accelerations and torques of the two bodies $\text{j1} = E(j, 1)$, and $\text{j2} = E(j, 2)$, that share the joint j. To preserve the total linear and angular momentum, the contribution is with alternating signs as

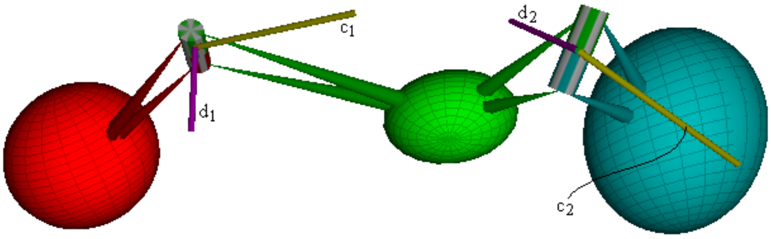
$$a_{\text{j1}} \mathrel{+}= m_{\text{j1}}^{-1}\,c_j \qquad \text{for } j = 1, 2, \ldots, n$$
$$a_{\text{j2}} \mathrel{-}= m_{\text{j2}}^{-1}\,c_j$$

and

$$\tau_{\text{j1}} \mathrel{+}= L_{j,\text{j1}}.R_{\text{j1}}^{-1}.c_j + Z_{j,\text{j1}}.R_{\text{j1}}^{-1}.d_j + R_{\text{j1}}^{-1}.e_j \qquad \text{for } j = 1, 2, \ldots, n$$
$$\tau_{\text{j2}} \mathrel{-}= L_{j,\text{j2}}.R_{\text{j2}}^{-1}.c_j + Z_{j,\text{j2}}.R_{\text{j2}}^{-1}.d_j + R_{\text{j2}}^{-1}.e_j$$

In total, the body i is subject to a linear acceleration a_i and torque τ_i of

(4) $$a_i = m_i^{-1}\,\Sigma_{j=1}^{n}\,\sigma_{j,i}\,c_j \qquad \text{for } i = 1, 2, \ldots, n+1$$
$$\tau_i = \Sigma_{j=1}^{n}\,\sigma_{j,i}(L_{j,i}.R_i^{-1}.c_j + Z_{j,i}.R_i^{-1}.d_j + R_i^{-1}.e_j)$$



Example: The linear accelerations and torques of the four bodies in the skeleton above are of the form

$$a_1 = m_1^{-1}(+c_1) \qquad \tau_1 = +L_{1,1}.R_1^{-1}.c_1 + Z_{1,1}.R_1^{-1}.d_1 + R_1^{-1}.e_1$$
$$a_2 = m_2^{-1}(-c_1 + c_2) \qquad \tau_2 = -L_{1,2}.R_2^{-1}.c_1 - Z_{1,2}.R_2^{-1}.d_1 - R_2^{-1}.e_1 + L_{2,2}.R_2^{-1}.c_2 + Z_{2,2}.R_2^{-1}.d_2 + R_2^{-1}.e_2$$
$$a_3 = m_3^{-1}(-c_2) \qquad \tau_3 = -L_{2,3}.R_3^{-1}.c_2 - Z_{2,3}.R_3^{-1}.d_2 - R_3^{-1}.e_2$$

Theorem: The linear acceleration a_i and torque τ_i for $i = 1, 2, \ldots, n+1$ as assigned in equations (4) result in the conservation of the total linear momentum, and total angular momentum of the skeleton. If all joints revolve frictionless and passively, i.e. $e_j = 0$ for all $j = 1, 2, \ldots, n$, the total kinetic energy is invariant, too.

Proof. The assignments (4) annihilate the time derivative d_t of the total linear momentum

$$\Sigma_{i=1}^{n+1} m_i\, a_i = \Sigma_{i=1}^{n+1} m_i\, m_i^{-1}\, \Sigma_{j=1}^{n} \sigma_{j,i}\, c_j = \Sigma_{j=1}^{n} \Sigma_{i=1}^{n+1} \sigma_{j,i}\, c_j = \Sigma_{j=1}^{n} c_j - \Sigma_{j=1}^{n} c_j = 0$$

and also the derivative d_t of the total angular momentum

$$\begin{aligned}
&\Sigma_{i=1}^{n+1}\; p_i \times (m_i\, a_i) + R_i.\Omega_i.I_i.\omega_i + R_i.I_i.\mathrm{d}_t\, \omega_i \\
&= \Sigma_{i=1}^{n+1}\; p_i \times (m_i\, a_i) + R_i.\Omega_i.I_i.\omega_i + R_i.I_i.[I_i^{-1}.(-\Omega_i.I_i.\omega_i + \tau_i)] \\
&= \Sigma_{i=1}^{n+1}\; p_i \times (m_i\, a_i) + R_i.\tau_i \\
&= \Sigma_{j=1}^{n}\; p_{j1} \times c_j + R_{j1}.[l_{j,j1} \times (R_{j1}^{-1}.c_j) + z_{j,j1} \times (R_{j1}^{-1}.d_j) + R_{j1}^{-1}.e_j] - \\
&\quad p_{j2} \times c_j - R_{j2}.[l_{j,j2} \times (R_{j2}^{-1}.c_j) + z_{j,j2} \times (R_{j2}^{-1}.d_j) + R_{j2}^{-1}.e_j] \\
&= \Sigma_{j=1}^{n}\; p_{j1} \times c_j + (R_{j1}.l_{j,j1}) \times c_j - p_{j2} \times c_j - (R_{j2}.l_{j,j2}) \times c_j + (R_{j1}.z_{j,j1} - R_{j2}.z_{j,j2}) \times d_j + e_j - e_j \\
&= \Sigma_{j=1}^{n} (p_{j1} + R_{j1}.l_{j,j1} - p_{j2} - R_{j2}.l_{j,j2}) \times c_j + 0 \times d_j + 0 \\
&= \Sigma_{j=1}^{n}\; 0 \times c_j \\
&= 0
\end{aligned}$$

making use of the relation $Q.[a \times (Q^{-1}.b)] = (Q.a) \times b$ for any (3)-vectors a, b and orthogonal (3,3)-matrix Q. The derivative d_t of the total kinetic energy simplifies to

$$\begin{aligned}
&\Sigma_{i=1}^{n+1} m_i\, v_i.a_i + \omega_i.I_i.\mathrm{d}_t\, \omega_i \\
&= \Sigma_{i=1}^{n+1} \Sigma_{j=1}^{n} m_i\, m_i^{-1}\, \sigma_{j,i}\, v_i.c_j + \omega_i.I_i.I_i^{-1}.(-\Omega_i.I_i.\omega_i + \sigma_{j,i}\, L_{j,i}.R_i^{-1}.c_j + \sigma_{j,i}\, Z_{j,i}.R_i^{-1}.d_j + \sigma_{j,i}\, R_i^{-1}.e_j) \\
&= \Sigma_{i=1}^{n+1} \Sigma_{j=1}^{n} \sigma_{j,i}\, v_i.c_j - \omega_i.\Omega_i.I_i.\omega_i + \sigma_{j,i}\, \omega_i.L_{j,i}.R_i^{-1}.c_j + \sigma_{j,i}\, \omega_i.Z_{j,i}.R_i^{-1}.d_j + \sigma_{j,i}\, \omega_i.R_i^{-1}.e_j \\
&= \Sigma_{j=1}^{n} \Sigma_{i=1}^{n+1} \sigma_{j,i} [v_i + \omega_i.L_{j,i}.R_i^{-1}].c_j + \sigma_{j,i}\, d_j.R_i.\Omega_i.z_{j,i} + \sigma_{j,i}\, e_j.R_i.\omega_i \\
&= \Sigma_{j=1}^{n} [v_{j1} + R_{j1}.\Omega_{j1}.l_{j,j1} - v_{j2} - R_{j2}.\Omega_{j2}.l_{j,j2}].c_j + d_j.[R_{j1}.\Omega_{j1}.z_{j,j1} - R_{j2}.\Omega_{j2}.z_{j,j2}] + e_j.[R_{j1}.\omega_{j1} - R_{j2}.\omega_{j2}] \\
&= \Sigma_{j=1}^{n}\; 0.\, c_j + d_j\, .0 + e_j.[R_{j1}.\omega_{j1} - R_{j2}.\omega_{j2}] \\
&= \Sigma_{j=1}^{n}\; e_j.[R_{j1}.\omega_{j1} - R_{j2}.\omega_{j2}]
\end{aligned}$$

because $\omega_i.\Omega_i = \omega_i \times \omega_i = 0$, $\omega_i.L_{j,i}.R_i^{-1} = -R_i.L_{j,i}.\omega_i = R_i.\Omega_i.l_{j,i}$, analogous, $\omega_i.Z_{j,i}.R_i^{-1} = R_i.\Omega_i.z_{j,i}$, $\Sigma_{i=1}^{n+1} \sigma_{j,i} = \sigma_{j,j1} + \sigma_{j,j2}$, and equations (2). If $e_j = 0$ for all $j = 1, 2, \ldots, n$, the total kinetic energy is invariant. ■

Remark: Each conservation law is granted by a different argument:

total linear momentum :	topology	$\sigma_{j,j1} + \sigma_{j,j2} = 0$
total angular momentum :	position of joint	equations (1)
total kinetic energy :	velocity of joint	equations (2) ■

At this point, the remaining issue is the computation of the vectors c_j, d_j for $j = 1, 2, \ldots, n$. To solve for these $6\,n$ unknowns, we substitute the terms (4) into equations (3). The vectors c_j, d_j are determined by a linear system of equations.

We convert the terms (4) into the notation of equations (3): From $a_i = m_i^{-1}\, \Sigma_{j=1}^{n} \sigma_{j,i}\, c_j$, we yield

$$a_{j1} = m_{j1}^{-1}\, \Sigma_{k=1}^{n} \sigma_{k,j1}\, c_k \qquad \text{for } j = 1, 2, \ldots, n$$
$$a_{j2} = m_{j2}^{-1}\, \Sigma_{k=1}^{n} \sigma_{k,j2}\, c_k$$

Further, we substitute $\tau_i = \Sigma_{j=1}^{n} \sigma_{j,i}[L_{j,i}.R_i^{-1}.c_j + Z_{j,i}.R_i^{-1}.d_j + R_i^{-1}.e_j]$ into the differential equation $\mathrm{d}_t\, \omega_i = I_i^{-1}.(-\Omega_i.I_i.\omega_i + \tau_i)$ and yield

$$\mathrm{d}_t\, \omega_{\mathrm{j1}} = I_{\mathrm{j1}}^{-1}.(-\Omega_{\mathrm{j1}}.I_{\mathrm{j1}}.\omega_{\mathrm{j1}} + \Sigma_{k=1}^{n} \sigma_{k,\mathrm{j1}}[L_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.c_k + Z_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.d_k + R_{\mathrm{j1}}^{-1}.e_k]) \qquad \text{for } j = 1, 2, \ldots, n$$
$$\mathrm{d}_t\, \omega_{\mathrm{j2}} = I_{\mathrm{j2}}^{-1}.(-\Omega_{\mathrm{j2}}.I_{\mathrm{j2}}.\omega_{\mathrm{j2}} + \Sigma_{k=1}^{n} \sigma_{k,\mathrm{j2}}[L_{k,\mathrm{j2}}.R_{\mathrm{j2}}^{-1}.c_k + Z_{k,\mathrm{j2}}.R_{\mathrm{j2}}^{-1}.d_k + R_{\mathrm{j2}}^{-1}.e_k])$$

The substitution of a_{j1}, and $\mathrm{d}_t\, \omega_{\mathrm{j1}}$ into the lhs of the equations (3) results in

$$(m_{\mathrm{j1}}^{-1}\, \Sigma_k\, \sigma_{k,\mathrm{j1}}\, c_k) + R_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.l_{j,\mathrm{j1}} - R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.(-\Omega_{\mathrm{j1}}.I_{\mathrm{j1}}.\omega_{\mathrm{j1}} + \Sigma_k\, \sigma_{k,\mathrm{j1}}[L_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.c_k + Z_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.d_k + R_{\mathrm{j1}}^{-1}.e_k])$$
$$= (m_{\mathrm{j1}}^{-1}\, \Sigma_k\, \sigma_{k,\mathrm{j1}}\, c_k) + R_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.l_{j,\mathrm{j1}} + R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.\Omega_{\mathrm{j1}}.I_{\mathrm{j1}}.\omega_{\mathrm{j1}} -$$
$$R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.\Sigma_k\, \sigma_{k,\mathrm{j1}}[L_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.c_k + Z_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.d_k + R_{\mathrm{j1}}^{-1}.e_k]$$
$$= [\Sigma_k\, \sigma_{k,\mathrm{j1}}\, ([m_{\mathrm{j1}}^{-1}\, \mathbf{1} - R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.L_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}].c_k - R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.Z_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.d_k - R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.R_{\mathrm{j1}}^{-1}.e_k)] +$$
$$R_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.l_{j,\mathrm{j1}} + R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.\Omega_{\mathrm{j1}}.I_{\mathrm{j1}}.\omega_{\mathrm{j1}}$$

and secondly

$$R_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.z_{j,\mathrm{j1}} - R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.(-\Omega_{\mathrm{j1}}.I_{\mathrm{j1}}.\omega_{\mathrm{j1}} + \Sigma_k\, \sigma_{k,\mathrm{j1}}[L_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.c_k + Z_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.d_k + R_{\mathrm{j1}}^{-1}.e_k])$$
$$= R_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.z_{j,\mathrm{j1}} + R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.\Omega_{\mathrm{j1}}.I_{\mathrm{j1}}.\omega_{\mathrm{j1}} - R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.\Sigma_k\, \sigma_{k,\mathrm{j1}}[L_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.c_k + Z_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.d_k + R_{\mathrm{j1}}^{-1}.e_k]$$
$$= [\Sigma_k\, \sigma_{k,\mathrm{j1}}(-R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.L_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.c_k - R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.Z_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1}.d_k - R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.R_{\mathrm{j1}}^{-1}.e_k)] +$$
$$R_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.z_{j,\mathrm{j1}} + R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.\Omega_{\mathrm{j1}}.I_{\mathrm{j1}}.\omega_{\mathrm{j1}}$$

with **1** as the identity (3,3)-matrix. Analogous, the substitution of a_{j2}, and $\mathrm{d}_t\, \omega_{\mathrm{j2}}$ into the rhs of (3) results in

$$[\Sigma_k\, \sigma_{k,\mathrm{j2}}\, ([m_{\mathrm{j2}}^{-1}\, \mathbf{1} - R_{\mathrm{j2}}.L_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.L_{k,\mathrm{j2}}.R_{\mathrm{j2}}^{-1}].c_k - R_{\mathrm{j2}}.L_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.Z_{k,\mathrm{j2}}.R_{\mathrm{j2}}^{-1}.d_k - R_{\mathrm{j2}}.L_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.R_{\mathrm{j2}}^{-1}.e_k)] +$$
$$R_{\mathrm{j2}}.\Omega_{\mathrm{j2}}.\Omega_{\mathrm{j2}}.l_{j,\mathrm{j2}} + R_{\mathrm{j2}}.L_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.\Omega_{\mathrm{j2}}.I_{\mathrm{j2}}.\omega_{\mathrm{j2}}$$

and secondly

$$[\Sigma_k\, \sigma_{k,\mathrm{j2}}([-R_{\mathrm{j2}}.L_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.L_{k,\mathrm{j2}}.R_{\mathrm{j2}}^{-1}].c_k - R_{\mathrm{j2}}.Z_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.Z_{k,\mathrm{j2}}.R_{\mathrm{j2}}^{-1}.d_k - R_{\mathrm{j2}}.Z_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.R_{\mathrm{j2}}^{-1}.e_k)] +$$
$$R_{\mathrm{j2}}.\Omega_{\mathrm{j2}}.\Omega_{\mathrm{j2}}.z_{j,\mathrm{j2}} + R_{\mathrm{j2}}.Z_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.\Omega_{\mathrm{j2}}.I_{\mathrm{j2}}.\omega_{\mathrm{j2}}$$

For $j, k = 1, 2, \ldots, n$, we define (6,6)-matrices

(5) $$A_{j,k,1} = \sigma_{k,\mathrm{j1}} \begin{pmatrix} R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.L_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1} - m_{\mathrm{j1}}^{-1}\, \mathbf{1} & R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.Z_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1} \\ R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.L_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1} & R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.Z_{k,\mathrm{j1}}.R_{\mathrm{j1}}^{-1} \end{pmatrix}$$
$$A_{j,k,2} = \sigma_{k,\mathrm{j2}} \begin{pmatrix} R_{\mathrm{j2}}.L_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.L_{k,\mathrm{j2}}.R_{\mathrm{j2}}^{-1} - m_{\mathrm{j2}}^{-1}\, \mathbf{1} & R_{\mathrm{j2}}.L_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.Z_{k,\mathrm{j2}}.R_{\mathrm{j2}}^{-1} \\ R_{\mathrm{j2}}.Z_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.L_{k,\mathrm{j2}}.R_{\mathrm{j2}}^{-1} & R_{\mathrm{j2}}.Z_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.Z_{k,\mathrm{j2}}.R_{\mathrm{j2}}^{-1} \end{pmatrix}$$

and (6)-vectors

$$b_{j,1} = \begin{pmatrix} R_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.l_{j,\mathrm{j1}} + R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.\Omega_{\mathrm{j1}}.I_{\mathrm{j1}}.\omega_{\mathrm{j1}} - \Sigma_k\, \sigma_{k,\mathrm{j1}}\, R_{\mathrm{j1}}.L_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.R_{\mathrm{j1}}^{-1}.e_k \\ R_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.\Omega_{\mathrm{j1}}.z_{j,\mathrm{j1}} + R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.\Omega_{\mathrm{j1}}.I_{\mathrm{j1}}.\omega_{\mathrm{j1}} - \Sigma_k\, \sigma_{k,\mathrm{j1}}\, R_{\mathrm{j1}}.Z_{j,\mathrm{j1}}.I_{\mathrm{j1}}^{-1}.R_{\mathrm{j1}}^{-1}.e_k \end{pmatrix}$$
$$b_{j,2} = \begin{pmatrix} R_{\mathrm{j2}}.\Omega_{\mathrm{j2}}.\Omega_{\mathrm{j2}}.l_{j,\mathrm{j2}} + R_{\mathrm{j2}}.L_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.\Omega_{\mathrm{j2}}.I_{\mathrm{j2}}.\omega_{\mathrm{j2}} - \Sigma_k\, \sigma_{k,\mathrm{j2}}\, R_{\mathrm{j2}}.L_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.R_{\mathrm{j2}}^{-1}.e_k \\ R_{\mathrm{j2}}.\Omega_{\mathrm{j2}}.\Omega_{\mathrm{j2}}.z_{j,\mathrm{j2}} + R_{\mathrm{j2}}.Z_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.\Omega_{\mathrm{j2}}.I_{\mathrm{j2}}.\omega_{\mathrm{j2}} - \Sigma_k\, \sigma_{k,\mathrm{j2}}\, R_{\mathrm{j2}}.Z_{j,\mathrm{j2}}.I_{\mathrm{j2}}^{-1}.R_{\mathrm{j2}}^{-1}.e_k \end{pmatrix}$$

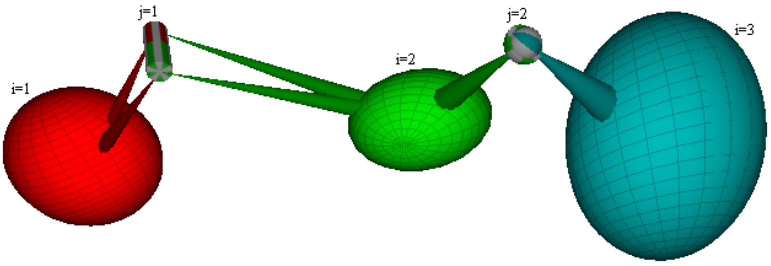
Then, equations (3) transform into $-\left[\Sigma_k\, A_{j,k,1}.\begin{pmatrix} c_k \\ d_k \end{pmatrix}\right] + b_{j,1} = -\left[\Sigma_k\, A_{j,k,2}.\begin{pmatrix} c_k \\ d_k \end{pmatrix}\right] + b_{j,2}$, or equivalently

$$\Sigma_{k=1}^{n}(A_{j,k,1} - A_{j,k,2}).\begin{pmatrix} c_k \\ d_k \end{pmatrix} = b_{j,1} - b_{j,2} \qquad \text{for } j = 1, 2, \ldots, n.$$

■ Introduction of spherical joints

To replace a hinge j by a spherical joint, we set the vector that represents the axis of the joint to zero, i.e. $\hat{z}_{j,1} = 0$ and $\hat{z}_{j,2} = 0$. Consequently, $z_{j,i} = 0$ for all $i = 1, 2, \ldots, n+1$. The equations of (1), (2), (3) that involve $z_{j,i}$ hold trivially.

Example: In the illustration below, joint $j = 1$ is a hinge, and joint $j = 2$ is a spherical joint. Therefore, we have $\hat{z}_{2,1} = 0$ and $\hat{z}_{2,2} = 0$.



■ Algorithm for animation

We describe how to 'integrate' the skeleton from time t over a time interval of length $h > 0$ to the next frame $t + h$. The input to the algorithm are the entities p_i, R_i, v_i, ω_i, I_i, m_i, and $L_{j,i}$, $Z_{j,i}$, e_j for all bodies $i = 1, 2, \ldots, n+1$ and joints $j = 1, 2, \ldots, n$ at time t, and the duration h. The values shall comply with the initial conditions (1) and (2). We determine the vectors c_j, d_j by solving the system of linear equations

$$(6) \qquad \begin{pmatrix} A_{1,1,1} - A_{1,1,2} & \cdots & A_{1,n,1} - A_{1,n,2} \\ \vdots & \ddots & \vdots \\ A_{n,1,1} - A_{n,1,2} & \cdots & A_{n,n,1} - A_{n,n,2} \end{pmatrix} . \begin{pmatrix} c_1 \\ d_1 \\ \vdots \\ c_n \\ d_n \end{pmatrix} = \begin{pmatrix} b_{1,1} - b_{1,2} \\ \vdots \\ b_{n,1} - b_{n,2} \end{pmatrix}$$

The terms $A_{j,k,1}$, $A_{j,k,2}$, $b_{j,1}$, and $b_{j,2}$ are defined in (5). Then, the linear accelerations a_i, and the torques τ_i are

$$a_i = m_i^{-1} \Sigma_{j=1}^{n} \sigma_{j,i}\, c_j \qquad \text{for } i = 1, 2, \ldots, n+1$$
$$\tau_i = \Sigma_{j=1}^{n} \sigma_{j,i} (L_{j,i}.R_i^{-1}.c_j + Z_{j,i}.R_i^{-1}.d_j + R_i^{-1}.e_j)$$

We reassign

$$v_i := v_i + a_i\, h \qquad \text{for } i = 1, 2, \ldots, n+1$$
$$p_i := p_i + v_i\, h$$
$$\omega_i := \omega_i + I_i^{-1}.(-\Omega_i.I_i.\omega_i + \tau_i)\, h$$
$$R_i := R_i.\exp[\Omega_i\, h]$$

The new values p_i, R_i, v_i, ω_i represent the dynamic configuration of the skeleton at time $t + h$ and are the output of the algorithm. (The (3,3)-matrix $\exp[\Omega_i\, h] = \sum_{k=0}^{\infty} (\Omega_i\, h)^k / (k!)$ is orthogonal.)

Remark: Equation (6) contains $6\,n$ unknowns: c_j, d_j for $j = 1, 2, \ldots, n$. However, the matrix is only of rank $5\,n_h + 3\,n_s$, where n_h is the number of hinges, n_s is the number of spherical joints, and $n_h + n_s = n$. If joint j is a hinge, the vector d_j is constrained to the plane orthogonal to the axis of the hinge $R_{j1}.z_{j1} = R_{j2}.z_{j2}$. If joint j is a spherical joint, then $d_j = 0$.

The solutions c_j, d_j of (6) are readily obtained via the pseudoinverse. An implementation of the singular value decomposition is stated in the book *Numerical Recipes in C++*, 2nd edition written by Press, Teucholsky, Vetterling, Flannery.

<u>Remark</u>: The substitution of $\sigma_{k,\text{j1}} = \delta_{E(j,1),E(k,1)} - \delta_{E(j,1),E(k,2)}$, and $\sigma_{k,\text{j2}} = \delta_{E(j,2),E(k,1)} - \delta_{E(j,2),E(k,2)}$ with

$$\delta_{i_1,i_2} = \begin{cases} 1 & \text{if } i_1 = i_2 \\ 0 & \text{otherwise} \end{cases}$$

transforms an entry of the block matrix of (6) into the alternate form

$$\begin{aligned} A_{j,k,1} - A_{j,k,2} = \\ +\delta_{E(j,1),E(k,1)} &\begin{pmatrix} R_{\text{j1}}.L_{j,\text{j1}}.I_{\text{j1}}^{-1}.L_{k,\text{j1}}.R_{\text{j1}}^{-1} - m_{\text{j1}}^{-1}\,\mathbf{1} & R_{\text{j1}}.L_{j,\text{j1}}.I_{\text{j1}}^{-1}.Z_{k,\text{j1}}.R_{\text{j1}}^{-1} \\ R_{\text{j1}}.Z_{j,\text{j1}}.I_{\text{j1}}^{-1}.L_{k,\text{j1}}.R_{\text{j1}}^{-1} & R_{\text{j1}}.Z_{j,\text{j1}}.I_{\text{j1}}^{-1}.Z_{k,\text{j1}}.R_{\text{j1}}^{-1} \end{pmatrix} \\ -\delta_{E(j,1),E(k,2)} &\begin{pmatrix} R_{\text{j1}}.L_{j,\text{j1}}.I_{\text{j1}}^{-1}.L_{k,\text{j2}}.R_{\text{j1}}^{-1} - m_{\text{j1}}^{-1}\,\mathbf{1} & R_{\text{j1}}.L_{j,\text{j1}}.I_{\text{j1}}^{-1}.Z_{k,\text{j2}}.R_{\text{j1}}^{-1} \\ R_{\text{j1}}.Z_{j,\text{j1}}.I_{\text{j1}}^{-1}.L_{k,\text{j2}}.R_{\text{j1}}^{-1} & R_{\text{j1}}.Z_{j,\text{j1}}.I_{\text{j1}}^{-1}.Z_{k,\text{j2}}.R_{\text{j1}}^{-1} \end{pmatrix} \\ -\delta_{E(j,2),E(k,1)} &\begin{pmatrix} R_{\text{j2}}.L_{j,\text{j2}}.I_{\text{j2}}^{-1}.L_{k,\text{j1}}.R_{\text{j2}}^{-1} - m_{\text{j2}}^{-1}\,\mathbf{1} & R_{\text{j2}}.L_{j,\text{j2}}.I_{\text{j2}}^{-1}.Z_{k,\text{j1}}.R_{\text{j2}}^{-1} \\ R_{\text{j2}}.Z_{j,\text{j2}}.I_{\text{j2}}^{-1}.L_{k,\text{j1}}.R_{\text{j2}}^{-1} & R_{\text{j2}}.Z_{j,\text{j2}}.I_{\text{j2}}^{-1}.Z_{k,\text{j1}}.R_{\text{j2}}^{-1} \end{pmatrix} \\ +\delta_{E(j,2),E(k,2)} &\begin{pmatrix} R_{\text{j2}}.L_{j,\text{j2}}.I_{\text{j2}}^{-1}.L_{k,\text{j2}}.R_{\text{j2}}^{-1} - m_{\text{j2}}^{-1}\,\mathbf{1} & R_{\text{j2}}.L_{j,\text{j2}}.I_{\text{j2}}^{-1}.Z_{k,\text{j2}}.R_{\text{j2}}^{-1} \\ R_{\text{j2}}.Z_{j,\text{j2}}.I_{\text{j2}}^{-1}.L_{k,\text{j2}}.R_{\text{j2}}^{-1} & R_{\text{j2}}.Z_{j,\text{j2}}.I_{\text{j2}}^{-1}.Z_{k,\text{j2}}.R_{\text{j2}}^{-1} \end{pmatrix} \end{aligned}$$